The BRST reduction of the chiral Hecke algebra

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Abstract

We explore the relationship between de Rham and Lie algebra cohomologies in the finite dimensional and affine settings. In particular, given a $\hat{\mathfrak{g}}_\kappa$-module that arises as the global sections of a twisted D-module on the affine flag manifold, we show how to compute its untwisted BRST reduction modulo $\mathfrak{n}(\mathcal{K})$ using the de Rham cohomology of the restrictions to $N(\mathcal{K})$ orbits. A similar relationship holds between the regular cohomology and the Iwahori orbits on the affine flag manifold. As an application of the above, we describe the BRST reduction of the chiral Hecke algebra as a vertex super algebra.

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1 Introduction.

A way of looking at geometric representation theory is as an attempt to match up algebraic objects that naturally arise in the study of representations of groups or algebras, with geometric objects which are perhaps easier to study. An early example of this is the Borel-Weil-Bott theorem that constructs irreducible representations of a reductive group via the sheaf cohomology of equivariant line bundles on the flag manifold of the group.

Expanding on the above approach, one may obtain representations of a Lie algebra $\mathfrak{g}$ by considering the global sections of a D-module on a homogeneous space of G, where $\mathfrak{g} = Lie(G)$. In the case of a reductive group G and its flag manifold G/B, we obtain in this way all representations of $\mathfrak{g}$ with the trivial central character. This is part of the work of Beilinson-Bernstein [3] which was

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aimed at proving the Kazhdan-Lusztig conjecture.[1] Note that G itself acts on both categories, via the twisting of $\mathfrak{g}$-modules by the adjoint action of $g \in G$ and the pullback of D-modules along the action of $g \in G$ on G/B. This identification is compatible with the actions, thus we can say that these two categories are but two incarnations of the correct analogue of the representations of G on the space of "functions" on G/B. For more on this point of view see [11].

A more complete version of the result of [3] is that representations of $\mathfrak{g}$ with other central characters may be obtained from appropriately twisted D-modules on G/B with the twisting corresponding to the central character. Thus the center of $U\mathfrak{g}$, i.e., the center of the enveloping algebra of $\mathfrak{g}$, serves as the space of "spectral parameters" for a decomposition of its category of representations. It coincides with the Bernstein center of the category of $\mathfrak{g}$-representations. This type of "spectral decomposition" of the category of $\mathfrak{g}$-representations and the identification of the "fibers" with categories of geometric origin has proven itself to be very useful.

Pursuing this further, we can also consider the setting of affine Kac-Moody Lie algebras $\hat{\mathfrak{g}}_\kappa$, which are infinite dimensional analogues of reductive Lie algebras. Here $\mathfrak{g}$ is as above and $\hat{\mathfrak{g}}_\kappa$ is a central extension[2] of the Lie algebra of the loop group $G(\mathcal{K})$ best thought of as parameterizing maps from the punctured formal disc $D^\times$ to G. The story becomes more interesting at this point and links up with the geometric Langlands program. Namely the space of "spectral parameters" now called local Langlands parameters is the moduli stack parameterizing de Rham $\check{G}$-local systems on $D^\times$ (i.e., $\check{G}$ principal bundles on $D^\times$ with an automatically flat connection), where we denote by $\check{G}$ the Langlands dual group of G.[3] Thus to each local Langlands parameter χ one must attach an appropriate subcategory $\hat{\mathfrak{g}}_\kappa - \mathrm{mod}_\chi$ of $\hat{\mathfrak{g}}_\kappa$-modules that is stable under the action of $G(\mathcal{K})$. Considerable progress has been made in this direction by Frenkel-Gaitsgory in the case of a critical level κ; it is well surveyed in [11]. The key aspect of this particular value of the level is that the center is very large. There is a conjecture of Beilinson, stated in the introduction to [2] that addresses the case of the negative integral level. The chiral Hecke algebra $S_\kappa(G)$ of Beilinson-Drinfeld plays a central role there.

In this paper we restrict our attention to the geometric part of the picture above. It fits into the general framework as follows: one considers only the subcategory of $\hat{\mathfrak{g}}_\kappa$-modules with "support" in the substack of regular singular connections on $D^\times$ with nilpotent residue; these have a conjectural interpretation as D-modules.[4] This substack can be described concretely as the quotient stack $\mathcal{N}/\check{G}$, where $\mathcal{N}$ is the nilpotent cone of $\check{G}$. Briefly, the elements of $\mathcal{N}$ represent the residue of the connection and the quotient by $\check{G}$ accounts for the gauge trans-

[1] The Kazhdan-Lusztig conjecture was independently demonstrated around the same time by Brylinski-Kashiwara in [9] using very similar methods.

[2] The central extensions are parameterized by the levels κ which are invariant inner products on the Cartan subalgebra (see Sec. 1.1 for more details). In this paper, outside of the introduction, we are only concerned with the negative integral level.

[3] A good reference for the notion of a category over a stack is [17].

[4] This is known as the tamely ramified case of the local geometric Langlands conjecture. Furthermore, we will focus almost exclusively on the unramified case.

formations. In fact the category of D-modules on $\mathcal{F}\ell$ is itself naturally a category over the stack $\mathcal{N}/\check{G}$ (this fact underlies [1, 7]). This explains the appearance of the regular singular condition on the connection.

Basically we want to emulate the correspondence between $D_{G/B}$-modules and $\mathfrak{g}$-modules with the trivial central character.[5] The role of D-modules is still played by D-modules, now on the affine flag manifold, however there is no obvious candidate for the subcategory of $\mathfrak{g}$-modules specified by the triviality of the central character, as the enveloping algebra of $\hat{\mathfrak{g}}_\kappa$, or rather its appropriate analogue, has no center. This corresponds to the fact that the moduli stack of de Rham local systems, discussed above, has no non-constant global functions [2]. The notion of "support" replaces the center, and the meaning to the "support" of $\hat{\mathfrak{g}}_\kappa$-modules is given through the consideration of the categories of modules over the twists of the chiral Hecke algebra by $\check{G}$-local systems. In the case under consideration, i.e. the analogue of the trivial central character, we look at local systems with regular singularities and nilpotent residue.[6]

In short, one wants to interpret D-modules on the affine flags as certain special $\hat{\mathfrak{g}}_\kappa$-modules just as in the Beilinson-Bernstein localization theorem. Currently there are partial analogues of the localization theorem in the context of the negative integral level. Namely, in the work of Beilinson-Drinfeld [5] and Frenkel-Gaitsgory [12] it is shown that the modules arising from appropriately twisted D-modules on either the affine flag manifold $\mathcal{F}\ell$ or the affine Grassmannian $\mathcal{G}r$ embed into the category of $\hat{\mathfrak{g}}_\kappa$-modules for the κ corresponding to the twisting. However, identifying the image of the above embedding is problematic. As mentioned above, the main candidate for the space of spectral parameters, namely the center of the enveloping algebra, that was used for this purpose in the finite dimensional case, is absent here. Instead one should, conjecturally, use the chiral Hecke algebra $S_\kappa(G)$. We postpone any discussion of $S_\kappa(G)$ to Sec. 4.1 and need only point out that $S_\kappa(G)$ is obtained from the twisted global sections of a D-module on the affine Grassmannian (denoted by $\widetilde{\mathcal{O}}_{\check{G}}$), it is $\check{G}$-equivariant (so that it can be twisted by a $\check{G}$-local system), and $S_\kappa(G)^{\check{G}}$ is the Kac-Moody vertex algebra $V_\kappa(\mathfrak{g})$ whose representation theory is the same as that of $\hat{\mathfrak{g}}_\kappa$.

The localization conjecture for the affine flag manifold that serves as one of the motivations for the present paper is an important special case of the conjecture outlined in the introduction of [2] (where this more general conjecture is settled for the considerably simpler commutative case). Namely, it is conjectured that there is an equivalence between, roughly speaking, the category of appropriately twisted equivariant representations of $S_\kappa(G)$ and the product of several copies of the cat-

[5] We point out at this time that what is actually accomplished here is only a first, albeit important, step.

[6] This is a general principle of generating representations of the smaller vertex algebra $V_\kappa(\mathfrak{g})$ by considering twisted representations of the larger $S_\kappa(G)$ that contains it. By keeping track of the twisting we obtain a measure of control over the representations of $V_\kappa(\mathfrak{g})$ that we allow. Here and below when speaking about $\hat{\mathfrak{g}}_\kappa$-modules we are implicitly using the fact that they are canonically identified with $V_\kappa(\mathfrak{g})$-modules, where $V_\kappa(\mathfrak{g})$ is the Kac-Moody vertex algebra.

egory of D-modules on $\mathcal{F}\ell$. More precisely, consider the following commutative diagram of functors

$$\begin{tikzcd}
(S_\kappa(G)_{\mathcal{N}}, \check{G}) - \mathrm{mod} \arrow[dr, "\Gamma(\mathcal{N},-)^{\check{G}}"'] & & D_{\mathcal{F}\ell} - \mathrm{mod} \arrow[ll, "F_\chi"'] \arrow[dl, "\Gamma(\mathcal{F}\ell,-\otimes\mathcal{L}_{\kappa+\chi})"] \\
& \hat{\mathfrak{g}}_\kappa - \mathrm{mod} &
\end{tikzcd}$$

and a few words of explanation for the symbols used are in order. As is repeatedly mentioned above, $S_\kappa(G)$ is a $\check{G}$-equivariant vertex algebra so that any $\check{G}$-local system ϕ on $D^\times$ gives rise to a new chiral algebra on the punctured disc that we denote by $S_\kappa(G)_\phi$. Recall that the moduli stack of $\check{G}$-local systems with a regular singularity and nilpotent residue is given by $\mathcal{N}/\check{G}$ so that we obtain in this way $S_\kappa(G)_{\mathcal{N}}$, a bundle of chiral algebras on $\mathcal{N}$ that is $\check{G}$-equivariant. We denote by $(S_\kappa(G)_{\mathcal{N}}, \check{G}) - \mathrm{mod}$ the category of $\check{G}$-equivariant $S_\kappa(G)_{\mathcal{N}}$-modules, with the notation for the other two categories being self explanatory. The functor F_χ is based on the concepts of [18] and [1]. It is roughly $\Gamma(\mathcal{F}\ell, (\mathcal{Z}(\mathcal{O}_{\check{G}}) \star -) \otimes \mathcal{L}_{\kappa+\chi})$ where $\mathcal{Z}$ is the functor from [18][7] and $\star$ is the fusion product, see [13, 18] for example. We say roughly because it is not clear how it lands in the $S_\kappa(G)_{\mathcal{N}}$-modules, to see this one needs some ideas of [1]. Let us mention that $G(\mathcal{K})$ acts on each category in the diagram and the functors commute with this action. There is an action of $Rep\,\check{G}$ on both sides of F_χ, obvious on the left and via $\mathcal{Z}$ on the right, and F_χ commutes with it.

The top arrow becomes (conjecturally) an equivalence of categories if we sum over appropriate representatives χ. Namely we consider the affine Weyl group dot action on the weight lattice of G, with respect to the level $\kappa - \kappa_c$, and χ is the only dominant regular, in the affine sense, weight in a given orbit.[8] The conjecture solves the problem of identifying precisely which $\hat{\mathfrak{g}}_\kappa$-modules come from D-modules on the affine flags: they are the ones that extend to an equivariant action of $S_\kappa(G)_{\mathcal{N}}$, the $\check{G}$-equivariant bundle of chiral algebras on $\mathcal{N}$ that contains $V_\kappa(\mathfrak{g})$.

The above is the tamely ramified case of the conjecture in [2]. Let us now consider the unramified case. It concerns the category $D^{m.a.}_{\mathcal{F}\ell} - \mathrm{mod}$ of monodromy annihilators, i.e. D-modules M on $\mathcal{F}\ell$ such that the monodromy action of [18] on $\mathcal{Z}(V)$, with V any representation of $\check{G}$, becomes trivial on $\mathcal{Z}(V) \star M$. When restricted to this subcategory the functor F_χ lands in $(S_\kappa(G), \check{G}) - \mathrm{mod}$ which is the subcategory of $(S_\kappa(G)_{\mathcal{N}}, \check{G}) - \mathrm{mod}$ supported at $0 \in \mathcal{N}$. This is because the lack of monodromy ensures that the action vertex operators of $S_\kappa(G)$ are no longer multi-valued and so we do not need to twist it by local systems in order to get rid of this complication. Thus we obtain the following diagram:

[7] This functor maps $Rep\,\check{G}$, the category of representations of $\check{G}$, to $D^I_{\mathcal{F}\ell} - \mathrm{mod}$, the category of Iwahori equivariant D-modules on $\mathcal{F}\ell$. Here $\mathcal{O}_{\check{G}}$ is the $\check{G}$-module of functions on $\check{G}$.

[8] This is the complementary point of view to our notion of sufficiently negative level, discussed in a Remark following Lemma 2.7.

$$\begin{tikzcd}
(S_\kappa(G), \check{G}) - \mathrm{mod} \arrow[dr, "-^{\check{G}}"'] & & D^{m.a.}_{\mathcal{F}\ell} - \mathrm{mod} \arrow[ll, "F_\chi"'] \arrow[dl, "\Gamma(\mathcal{F}\ell, - \otimes \mathcal{L}_{\kappa+\chi})"] \\
& \hat{\mathfrak{g}}_\kappa - \mathrm{mod} &
\end{tikzcd}$$

and the conjecture is that F_χ is an equivalence after summing over χ as above.

It would be interesting to investigate the relationship between $D_{\mathcal{G}r} - \mathrm{mod}$ and $D^{m.a.}_{\mathcal{F}\ell} - \mathrm{mod}$. Namely as seen in the following diagram:

$$\begin{tikzcd}
(S_\kappa(G)_{\mathcal{N}}, \check{G}) - \mathrm{mod} & D_{\mathcal{F}\ell} - \mathrm{mod} \arrow[l, "F_{2\rho}"'] \\
(S_\kappa(G), \check{G}) - \mathrm{mod} \arrow[u, "i_\bullet"] & D_{\mathcal{G}r} - \mathrm{mod} \arrow[l, "\Gamma(\mathcal{G}r, (\widetilde{\mathcal{O}}_{\check{G}} \star -) \otimes \mathcal{L}_\kappa)"'] \arrow[u, "\pi^*"']
\end{tikzcd}$$

the D-modules from $\mathcal{G}r$ provide a large supply of monodromy annihilators via π^*. However it is not difficult to come up with, using [1], examples of m.a. D-modules that are not pulled back from $\mathcal{G}r$, at least not via π^*.[9] One may naively conjecture, based on a similar result [15] on the level of derived categories, that the category $D^{m.a.}_{\mathcal{F}\ell} - \mathrm{mod}$ is obtained from $D_{\mathcal{G}r} - \mathrm{mod}$ via base change from the stack $0/\check{G}$ to $0/\check{B}$. This would have an interesting consequence that a $\check{G}$-equivariant $S_\kappa(G)$-module can be twisted not only by a $\check{G}$-representation, but more generally, by a $\check{B}$-representation.

Another question is how to describe the image of $F_{2\rho} \circ \pi^*$ above. A possible answer is discussed after Corollary 3.8 and involves the BRST functor. The key is that the BRST reduction of an $S_\kappa(G)$-module that comes directly from a D-module on the affine Grassmannian has a very compact and conjecturally characterizing form in terms of the de Rham cohomology of the restrictions of the original D-module to the semi-infinite orbits in $\mathcal{G}r$.

In this paper we compute by a mixture of algebraic and geometric methods, $H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), S_\kappa(G))$, i.e., the semi-infinite cohomology of $\mathfrak{n}(\mathcal{K})$ with coefficients in $S_\kappa(G)$. It follows from general considerations that as $S_\kappa(G)$ is a vertex algebra, so is $H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), S_\kappa(G))$ and we explicitly describe its vertex algebra structure.

For a $\hat{\mathfrak{g}}_\kappa$-module M, the motivation for considering its BRST reduction, as $H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), M)$ is called, lies in the suggestive fact that it is a $\hat{\mathfrak{h}}_{\kappa-\kappa_c}$-module. Thus the problem shifts from the domain of the non-commutative G to the more accessible case of its commutative torus H.[10] Broadly described, a possible approach to the problem of identifying $D^{m.a.}_{\mathcal{F}\ell} - \mathrm{mod}$ with $(S_\kappa(G), \check{G}) - \mathrm{mod}$ consists of first trying to enumerate the images of the objects on both sides under appropriate functors and then hope to lift this identification to the original categories.

[9] For $G = PGL_2$ there is an automorphism σ of $\mathcal{F}\ell$ such that $\sigma^*\pi^*$ also produces monodromy annihilators.

[10] In fact in [2] the conjecture is checked for the case of $G = H$.

The functors are, to first approximation, $\oplus_{w\in W_{\mathrm{aff}}} H^\bullet_{DR}(S_w, i_w^! -)$ on the D-module side[11] and the BRST reduction on the $S_\kappa(G)$ side.[12] The latter requires a few words of explanation. If M is a $\check{G}$-equivariant $S_\kappa(G)$-module then by the results of this paper $H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), M)$ is a $\check{G}$-equivariant module over the vertex algebra of global sections of a $\check{G}$-equivariant bundle of vertex algebras over $\check{G}/\check{H}$. The category of such modules is then equivalent to the category of $\check{H}$-equivariant modules over the vertex algebra that is the fiber of the bundle above over $\check{H} \in \check{G}/\check{H}$. This fiber is an enlargement of the much studied lattice Heisenberg vertex algebra,[13] in fact the lattice Heisenberg vertex algebra is exactly its cohomological degree 0 part.[14] The representation theory of the lattice Heisenberg is well understood; it has a finite number of irreducible modules. This observation agrees with the fact that one should really consider a product of several copies of $D_{\mathcal{F}\ell} - \mathrm{mod}$ as corresponding to $(S_\kappa(G), \check{G}) - \mathrm{mod}$. Recalling now that we still have the $\check{H}$-grading and the remaining part of the fiber vertex algebra, we see that one has roughly the same type of object as $\oplus_{w\in W_{\mathrm{aff}}} H^\bullet_{DR}(S_w, i_w^! M)$. We hope that the results and methods of this paper will provide a way to illuminate the relationship between the algebraic, i.e., the representation theoretic side and the geometric, i.e., the D-module side of the above conjectural correspondence.

This text is organized as follows. Section 2 contains comparison theorems between Lie algebra and De Rham cohomologies that we will subsequently need. Some of the results in this section (in particular the ones pertaining to the finite dimensional situation) are believed to be part of the folklore; unfortunately we cannot cite a reference other than this text. The proofs provided here are based on A. Voronov's semi-infinite induction (alternatives are demonstrated in the finite case). It is worth noting that Theorems 2.1 and 2.5 are essentially the same, but the proofs illustrate very different approaches. They address the finite-dimensional case. Theorem 2.9 is the main theorem of this section, it deals with the semi-infinite version of the infinite dimensional case.

Section 3 is devoted to the computation of the BRST reduction of the chiral Hecke algebra, first as a module over the Heisenberg Lie algebra (Corollary 3.2), and finally, in the main theorem of the paper (Theorem 3.7), as a vertex algebra. Some of the ingredients used in the proof are Theorem 2.9 and the Mirković-Vilonen theorem [21, 22].

In section 4 we provide some auxiliary information that the reader should find useful. Namely, a brief overview of the Beilinson-Drinfeld chiral Hecke algebra is included (Sec. 4.1). No references containing a construction were available for citation, however a brief discussion can be found in [10]. The language of the highest weight algebras is introduced (Sec. 4.2) as it is useful for stating the main results of the paper. Also included in Sec. 4.2 are certain details on how the Heisenberg Lie algebra module structure on a vertex algebra determines the vertex

[11] The S_w are the $N(\mathcal{K})$-orbits which are labeled by the elements of the affine Weyl group W_{aff}.
[12] This is illustrated by Corollaries 3.8 and 3.9.
[13] Coincidentally, the lattice Heisenberg vertex algebra is the chiral Hecke algebra for $G = H$.
[14] The remaining part is roughly $H^\bullet(\mathfrak{n}, \mathbb{C})$.

operators modulo the knowledge of the highest weight algebra.

Some sources containing the background material for this paper that we recommend are [6, 16, 10]. Finally, the terms semi-infinite cohomology and BRST reduction are used interchangeably and we refer the reader to [25] for the definitions. A sketch of the relevant details is given in the discussion preceding Lemma 2.7. The matter of notation is addressed below. Since our sources do not use mutually compatible notation, we made some choices that are to the best of our knowledge consistent.

1.1 Some notational conventions.

We encourage the reader to quickly skim this section and to review whenever necessary.

The group G that we consider is a simple algebraic group over $\mathbb{C}$, and $\mathfrak{g}$ is its Lie algebra. Some of the groups and algebras that we need are the Lie algebras $\mathfrak{b} \subset \mathfrak{g}$, the Borel subalgebra, $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, the nilpotent subalgebra, and $\mathfrak{h} = \mathfrak{b}/\mathfrak{n}$ the Cartan Lie algebra; the corresponding groups are denoted by B, N, and H. We reserve $\mathfrak{b}^-$, $\mathfrak{n}^-$, etc for the opposite versions, i.e., $\mathfrak{n}^-$ is the sum of the negative root spaces. Note that $\mathfrak{h}$ is sometimes used to denote a subalgebra of $\mathfrak{b}$ but this requires a choice, the same holds for H.

Put $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. Let $\mathfrak{g}(\mathcal{K}) = \mathfrak{g}\hat{\otimes}\mathcal{K}$, and define $\mathfrak{g}(\mathcal{O})$, $\mathfrak{n}(\mathcal{K})$, and $\mathfrak{n}(\mathcal{O})$ similarly. Denote by $G(\mathcal{K})$ and $N(\mathcal{K})$ the algebraic loop groups of G and N, by $G(\mathcal{O})$ and $N(\mathcal{O})$ the subgroups of positive loops. Denote by $\mathfrak{n}(\mathcal{K})_\dagger$ and $\mathfrak{n}(\mathcal{O})_\dagger$, $\mathfrak{n}(\mathcal{K}) \oplus t\mathfrak{h}(\mathcal{O})$ and $\mathfrak{n}(\mathcal{O}) \oplus t\mathfrak{h}(\mathcal{O})$ respectively.

Given an invariant inner product $(\cdot,\cdot)_\kappa$ on $\mathfrak{g}$, the affine Kac-Moody Lie algebra $\hat{\mathfrak{g}}_\kappa$ is defined as the central extension $\mathfrak{g}(\mathcal{K})^\sim$ of $\mathfrak{g}(\mathcal{K})$, with the cocycle ϕ given by $\phi(x \otimes f, y \otimes g) = -(x, y)_\kappa \mathrm{Res} f dg$, where $x, y \in \mathfrak{g}$ and $f, g \in \mathcal{K}$. For the purposes of this paper $(\cdot,\cdot)_\kappa = \kappa(\cdot,\cdot)_0$ with $\kappa < -h^\vee$, where $(\cdot,\cdot)_0$ is the normalized invariant inner product on $\mathfrak{g}$ (i.e., $(\theta,\theta)_0 = 2$, where θ is the highest weight of the adjoint representation) and $h^\vee = 1 + (\rho, \theta)_0$ ($\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$) the dual Coxeter number of $\mathfrak{g}$. This ensures that the level super line bundle is defined and twisting by it makes the global sections functor exact and faithful [12]. We note that $(\cdot,\cdot)_{\kappa_c} = -\frac{1}{2}(\cdot,\cdot)_{Kil} = -h^\vee(\cdot,\cdot)_0$.

Let Γ denote the co-weight lattice and $\check{\Gamma}$ the weight lattice of G. Write $\mathcal{L}_\chi$ for the line bundle with total space $G \times_B \mathbb{C}_{-\chi}$ for $\chi \in \check{\Gamma}$. We point out that for us $\mathbb{C}_\chi$ denotes a non-trivialized line on which $\mathfrak{h}$ (or $\check{\mathfrak{h}}$) acts via the character (or co-character) χ. As usual W and W_{aff} denote the Weyl and the affine Weyl groups respectively, note that $W_{\mathrm{aff}} = \Gamma \rtimes W$. In the finite setting the dot action of W is defined by $w \cdot \chi = w(\chi + \rho) - \rho$, where ρ is the half sum of the positive roots. In the affine setting the dot action depends on the level κ and for $w \in W_{\mathrm{aff}}$ with $w = \lambda_w \bar{w}$, is given by $w \cdot \chi = \bar{w} \cdot \chi - \kappa(\lambda_w)$.

To emphasize the role of ρ we use the convention that χ is called dominant if $(\chi + \rho)(H_\alpha) \notin \{-1, -2, -3, ...\}$ for each positive coroot H_α. We say that χ is

dominant regular if $\chi - \rho$ is dominant. We note that it is very common to call the latter dominant, we do not follow that convention.

Denote by I the Iwahori subgroup of $G(\mathcal{O})$, more precisely, $I = \mathrm{ev}^{-1}(B)$ where $\mathrm{ev} : G(\mathcal{O}) \to G$ is the usual evaluation map. Set $I^+ = \mathrm{ev}^{-1}(N)$. We will use $\mathfrak{i}$, $\mathfrak{i}^+$ for $Lie(I)$, $Lie(I^+)$ respectively. Let $\mathcal{F}\ell$ denote the affine flag manifold and $\mathcal{G}r$ the affine Grassmannian, roughly speaking $\mathcal{F}\ell = G(\mathcal{K})/I$ and $\mathcal{G}r = G(\mathcal{K})/G(\mathcal{O})$.

We reserve $\mathcal{T}_X$ and $\mathcal{O}_X$ for the sheaves of vector fields and functions on X respectively. If $C^\bullet$ is a complex, then $C^\bullet[n]$ denotes a degree shift, i.e., the degree k component of $C^\bullet[n]$ is C^{k+n}.

2 Lie algebra and De Rham cohomologies.

We are interested in reducing the Lie algebra cohomology (usual or semi-infinite) computations for modules that arise geometrically as twisted global sections of a D-module on a certain G-space, to the computation of the De Rham cohomology of the D-module itself restricted to orbits. We begin with the motivational finite-dimensional setting and proceed to the case of interest, the affine setting.

2.1 The finite-dimensional setting.

Let X be an homogeneous G space. Then by differentiating the G action we obtain a map of Lie algebras $\alpha : \mathfrak{g} \longrightarrow \Gamma(X, \mathcal{T}_X)$, which after taking the dual gives $\Gamma(X, \Omega^i_X) \longrightarrow \bigwedge^i \mathfrak{g}^* \otimes \Gamma(X, \mathcal{O}_X)$. Furthermore if M is a left[15] D-module on X, then $\Gamma(X, M)$ is a $\mathfrak{g}$ module, and we have a map $\Gamma(X, M \otimes \Omega^\bullet_X) \longrightarrow \bigwedge^\bullet \mathfrak{g}^* \otimes \Gamma(X, M)$. If X is affine, the complex on the left computes $H^\bullet_{DR}(X, M)$, while the one on the right computes $H^\bullet(\mathfrak{g}, \Gamma(X, M))$, and as our map commutes with the differentials, it descends to the cohomology, namely

$$\alpha^* : H^\bullet_{DR}(X, M) \longrightarrow H^\bullet(\mathfrak{g}, \Gamma(X, M)).$$

In addition, if the action of G on X extends to that of G' in which G is normal then both sides above are $\mathfrak{g}'/\mathfrak{g}$-modules and the map is compatible with this action. Note that $H^\bullet_{DR}(X, M)$ is a trivial $\mathfrak{g}'/\mathfrak{g}$-module. Furthermore, in the case when X is a G torsor α^* is an isomorphism even on the level of complexes.

Let us apply this observation to the following situation. Given a D-module M on G/B, one may consider $H^\bullet(\mathfrak{n}, \Gamma(G/B, M \otimes \mathcal{L}_\chi))$ as a $\mathfrak{h}$-module. We should immediately restrict our attention to χ dominant regular[16] as this ensures the exactness of $\Gamma(G/B, - \otimes \mathcal{L}_\chi))$. In that case we have the following:

[15] All D-modules in the finite setting are left by default, though we consider the right ones in Theorem 2.5. In the affine setting, only the right D-modules exist.

[16] See the remark following Theorem 2.1 for the non dominant regular χ case.

Theorem 2.1. *Let M be a D-module on G/B, and $X_w \subset G/B$ the N orbit labeled by $w \in W$, let χ be dominant regular, then as $\mathfrak{h}$-modules*

$$H^\bullet(\mathfrak{n}, \Gamma(G/B, M \otimes \mathcal{L}_\chi)) \cong \bigoplus_{w \in W} H^\bullet_{DR}(X_w, i_w^! M) \otimes \mathbb{C}_{w\cdot(-2\rho-\chi)}.$$

Remark. The proof given below, while illuminating, is ultimately a digression. The reader may skip to Theorem 2.5 which, along with its proof, is a baby version of the one in the affine setting.

Proof. Recall that we have a notion of length for the elements w of the Weyl group W, in particular the length $\ell(w)$ is equal to the dimension of the corresponding N-orbit X_w. We observe that G/B has a filtration (see the next paragraph) $S_i = \coprod_{\ell(w)\leq i} X_w$ which equips M with a filtration in the derived category with associated graded factors $i_{w*}i_w^! M$. Applying $\Gamma(G/B, - \otimes \mathcal{L}_\chi))$, we get a filtration on $\Gamma(G/B, M \otimes \mathcal{L}_\chi)$. This reduces the theorem to the special case of $M = i_{w*}M_0$ as the $\mathfrak{h}$ action on the cohomology of the factors is then different for different w's and so the spectral sequence degenerates and $H^\bullet(\mathfrak{n}, \Gamma(G/B, M \otimes \mathcal{L}_\chi))$ is canonically isomorphic to $\bigoplus_{w\in W} H^\bullet(\mathfrak{n}, \Gamma(G/B, i_{w*}i_w^! M \otimes \mathcal{L}_\chi))$.

Since the above type of argument is used repeatedly in the rest of the paper we provide some additional details. Consider a decomposition of a space $Y = \coprod Z_i$ with $T_n = \coprod_{i\leq n} Z_i$ closed in Y. Let $\iota_n : T_n \hookrightarrow Y$, $\alpha_n : T_{n-1} \hookrightarrow T_n$, $j_n : Z_n \hookrightarrow T_n$ and $i_n : Z_n \hookrightarrow Y$ so that $i_n = \iota_n \circ j_n$ and $\iota_{n-1} = \iota_n \circ \alpha_n$. If M is a D-module on Y then by considering $\iota_n^! M$ on T_n and the decomposition $T_n = T_{n-1} \coprod Z_n$ we obtain a distinguished triangle in the derived category $\alpha_{n*}\alpha_n^! \iota_n^! M \to \iota_n^! M \to j_{n*}j_n^! \iota_n^! M$. If we apply ι_{n*} to it we get another distinguished triangle

$$\iota_{(n-1)*}\iota_{n-1}^! M \to \iota_{n*}\iota_n^! M \to i_{n*}i_n^! M.$$

The filtration on M is thus given by $\iota_{n*}\iota_n^! M$ with the associated graded factors $i_{n*}i_n^! M$. Since $\Gamma(G/B, - \otimes \mathcal{L}_\chi)$ is exact by [3], it preserves distinguished triangles and applying the cohomological functor $H^\bullet(\mathfrak{n}, -)$ we obtain the desired spectral sequence. In fact one can avoid any reference to the machinery of spectral sequences by using induction and long exact sequences that will be canonically split exact using the $\mathfrak{h}$ action.

We are ready to proceed, begin with w_0, the longest element in W, i.e., the element corresponding to the big cell in G/B. Dropping the subscript in M_0, we have

$$\begin{aligned} H^\bullet(\mathfrak{n}, \Gamma(G/B, i_{w_0*}M \otimes \mathcal{L}_\chi)) &\cong H^\bullet(\mathfrak{n}, \Gamma(X_w, M \otimes \mathcal{L}_\chi|_{X_{w_0}})) \\ &\cong H^\bullet(\mathfrak{n}, \Gamma(X_w, M)) \otimes \mathbb{C}_{w_0(-\chi)} \\ &\cong H^\bullet_{DR}(X_{w_0}, M) \otimes \mathbb{C}_{w_0(-\chi)}, \end{aligned}$$

the last step follows from the discussion above as X_{w_0} is an N torsor. Note that we do not need χ to be dominant regular here.

To prove the theorem for other $w \in W$ we reduce to the case of w_0 using the following observation. Let Y_w in $G/B \times G/B$ be the G orbit through (B, wB), denote by p_1 and p_2 the restriction to Y_w of the projections onto the factors. For M a D-module on G/B, set $\widetilde{M}^w = p_{2*}p_1^*M$, then

$$\Gamma(G/B, M \otimes \mathcal{L}_\chi) \cong R\Gamma(G/B, \widetilde{M}^w \otimes \mathcal{L}_{w^{-1}\cdot\chi})$$

as $\mathfrak{g}$-modules. Let us suppress the exponent in $\widetilde{M}^w$ once it is established which w we are using.

Now let M be a D-module on X_w. Consider the diagram

$$\begin{tikzcd} G/B & Y_{w^{-1}w_0} \arrow[l, "p_1"'] \arrow[r, "p_2"] & G/B \\ X_w \arrow[u, "i_w"] & Y'_{w^{-1}w_0} \arrow[l, "p'_1"'] \arrow[r, "p'_2"] \arrow[u, "i"] & X_{w_0} \arrow[u, "i_{w_0}"] \end{tikzcd}$$

where the left square above is Cartesian by definition (i.e., Y' is defined by the diagram itself), p'_1 has affine space fibers, and p'_2 is an isomorphism. So that $p_{2*}p_1^*i_{w*}M \cong p_{2*}i_*p_1'^*M \cong i_{w_0*}p'_{2*}p_1'^*M$, hence $i^!_{w_0}\widetilde{i_{w*}M}^{w^{-1}w_0} \cong p'_{2*}p_1'^*M$. The proof is then completed by the following chain of isomorphisms:

$$\begin{aligned} H^\bullet(\mathfrak{n}, \Gamma(G/B, i_{w*}M \otimes \mathcal{L}_\chi)) &\cong H^\bullet(\mathfrak{n}, R\Gamma(G/B, \widetilde{i_{w*}M} \otimes \mathcal{L}_{w_0w\cdot\chi})) \\ &\cong H^\bullet(\mathfrak{n}, \Gamma(X_{w_0}, i^!_{w_0}\widetilde{i_{w*}M})) \otimes \mathbb{C}_{w\cdot(-2\rho-\chi)} \\ &\cong H^\bullet_{DR}(X_{w_0}, i^!_{w_0}\widetilde{i_{w*}M}) \otimes \mathbb{C}_{w\cdot(-2\rho-\chi)} \\ &\cong H^\bullet_{DR}(X_{w_0}, p'_{2*}p_1'^*M) \otimes \mathbb{C}_{w\cdot(-2\rho-\chi)} \\ &\cong H^\bullet_{DR}(Y'_{w^{-1}w_0}, p_1'^*M) \otimes \mathbb{C}_{w\cdot(-2\rho-\chi)} \\ &\cong H^\bullet_{DR}(X_w, M) \otimes \mathbb{C}_{w\cdot(-2\rho-\chi)}. \end{aligned}$$

□

Remark. The assumption that χ be dominant regular is necessary, however there is a way to replace M by $\widetilde{M}^w$, very similar to the method used in the proof of Theorem 2.1 in such a way that we have for $w^{-1}\cdot\chi$ dominant regular

$$R\Gamma(G/B, M \otimes \mathcal{L}_\chi) \cong \Gamma(G/B, \widetilde{M}^w \otimes \mathcal{L}_{w^{-1}\cdot\chi}).$$

When χ isn't dominant regular but $w^{-1}\cdot\chi$ is[17], this reduces the problem to our familiar case. The construction of $\widetilde{M}^w$ is immediate from the observation that for any character χ, we have that $\Gamma(G/B, i_{e*}\mathcal{O}_e \otimes \mathcal{L}_\chi)$ is the Verma module with highest weight $-2\rho - \chi$, while for χ dominant regular $\Gamma(G/B, i_{w!}\mathcal{O}_w \otimes \mathcal{L}_\chi)$ is the Verma module with highest weight $w\cdot(-2\rho-\chi)$. Explicitly we set $\widetilde{M}^w = p_{2!}p_1^*M$,

[17] Such a $w \in W$ exists if and only if $\langle\check{\alpha}, \chi + \rho\rangle \neq 0$ for all $\alpha \in R$.

where p_1 and p_2 are as in the proof of Theorem 2.1. This "intertwining functors" construction originates in [4].

So far we have been using left D-modules implicitly, however in the affine setting only right D-modules exist, and so we switch to using them exclusively at this point. Furthermore the proof of Theorem 2.1 does not immediately generalize to that setting. It was included because its very geometric nature appealed to us. Now we must switch to a more algebraic approach that directly generalizes. We begin with some preliminaries.

The following is a version of the Shapiro Lemma[18]. Note that $\mathfrak{g}$ must be finite dimensional. For a finite dimensional V, we use $\det(V)$ to denote its top exterior power $\wedge^{\dim(V)}V$; it is a non-trivialized line.

Lemma 2.2. *Let $\mathfrak{k} \subset \mathfrak{g}$ be a Lie subalgebra, then there is a natural isomorphism:*

$$H^\bullet(\mathfrak{k}, M \otimes det(\mathfrak{g}/\mathfrak{k})^*) \xrightarrow{\sim} H^\bullet(\mathfrak{g}, Ind_{\mathfrak{k}}^{\mathfrak{g}} M)[dim\mathfrak{g} - dim\mathfrak{k}]$$

where M is a $\mathfrak{k}$-module.

Proof. If L is a finite dimensional Lie algebra and N an L-module, then there is an isomorphism

$$H^\bullet(L, N \otimes \det(L)) \xrightarrow{\sim} H_\bullet(L, N)[-\dim(L)]$$

given by the contraction of $\det(L)$ with forms $\omega \in \wedge^\bullet L^*$. One checks that the map commutes with the differentials and it is clearly an isomorphism on the level of complexes. There is a map in the other direction obtained by

$$H_\bullet(L, N) = H_\bullet(L, (N \otimes \det(L)) \otimes \det(L^*)) \to H^\bullet(L, N \otimes \det(L))[\dim(L)]$$

similarly through contraction. The following chain of isomorphisms completes the proof:

$$\begin{aligned} H^\bullet(\mathfrak{k}, M \otimes \det(\mathfrak{g}/\mathfrak{k})^*) &\xrightarrow{\sim} H_\bullet(\mathfrak{k}, M \otimes \det(\mathfrak{g}/\mathfrak{k})^* \otimes \det(\mathfrak{k}^*))[-\dim(\mathfrak{k})] \\ &\cong H_\bullet(\mathfrak{k}, M \otimes \det(\mathfrak{g}^*))[-\dim(\mathfrak{k})] \end{aligned}$$

by Shapiro Lemma

$$\xrightarrow{\sim} H_\bullet(\mathfrak{g}, \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(M \otimes \det(\mathfrak{g}^*)))[-\dim(\mathfrak{k})]$$

By universality there is a map of $\mathfrak{g}$-modules $\mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(M \otimes \det(\mathfrak{g}^*)) \to \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(M) \otimes \det(\mathfrak{g}^*)$ that is compatible with the natural filtration on the modules and is an isomorphism on the associated graded pieces. Thus it is an isomorphism of modules:

$$\begin{aligned} &\xrightarrow{\sim} H_\bullet(\mathfrak{g}, \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(M) \otimes \det(\mathfrak{g}^*))[-\dim(\mathfrak{k})] \\ &\xrightarrow{\sim} H^\bullet(\mathfrak{g}, \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}} M)[\dim\mathfrak{g} - \dim\mathfrak{k}]. \end{aligned}$$

[18] It has a semi-infinite analogue [24].

Note that the map of the Lemma can be written down explicitly as follows. Observe that $\det(\mathfrak{g}/\mathfrak{k})^*$ is naturally a line in $\wedge^\bullet\mathfrak{g}^*$ and whereas there is no canonical map from $\wedge^\bullet\mathfrak{k}^*$ to $\wedge^\bullet\mathfrak{g}^*$, there is one from $\wedge^\bullet\mathfrak{k}^*\otimes\det(\mathfrak{g}/\mathfrak{k})^*$ to $\wedge^\bullet\mathfrak{g}^*[\dim\mathfrak{g}-\dim\mathfrak{k}]$. Tensoring this map with $M\hookrightarrow \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}M$ yields the required isomorphism. □

By a (right) D-module of delta functions at x in X we mean the D-module $i_{x*}\mathbb{C}$ where $i_x:\{x\}\hookrightarrow X$; we denote it by δ_x. If $x=B\in G/B$ then the sections of δ_x (as a $\mathfrak{g}$-module) can be described explicitly as $U\mathfrak{g}/U\mathfrak{b}=\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}\mathbb{C}$. Recall that such an object is called a Verma module. In the representation theory of $\mathfrak{g}$ one also has a Co-Verma module $\mathrm{Coind}_{\mathfrak{b}^-}^{\mathfrak{g}}\mathbb{C}$ and everything in between called semi-induced modules [24, 25]. The precise definition of the semi-induced module is not straightforward, however what we need is the fact that all of these $\mathfrak{g}$-modules have the same character, i.e., agree as $\mathfrak{h}$-modules. At least as $\mathfrak{n}$-modules they can be constructed through co-induction followed by induction (see the proof of the Lemma below). Furthermore, each is well adapted to a particular (co)homology theory. More precisely, $H_\bullet(\mathfrak{n}^-,\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}\mathbb{C})=\mathbb{C}$, $H^\bullet(\mathfrak{n},\mathrm{Coind}_{\mathfrak{b}^-}^{\mathfrak{g}}\mathbb{C})=\mathbb{C}$, etc.

When we consider appropriately twisted delta functions, i.e., $\delta_x\otimes\mathcal{L}_\chi$ for χ sufficiently dominant, then (see [20]) the $\mathfrak{g}$-module $\Gamma(G/B,\delta_x\otimes\mathcal{L}_\chi)$ is simple and thus its cohomology may be computed by identifying it with any one of the (isomorphic in this case) semi-induced modules. This is the idea behind the proof of the Lemma below as well as its affine analogues.

Lemma 2.3. *Let δ_x be the right D-module of delta functions at $x\in G/B$ and $\chi-2\rho$ be dominant regular, then*

$$H^\bullet(\mathfrak{n},\Gamma(G/B,\delta_x\otimes\mathcal{L}_\chi))\cong\mathcal{L}_\chi|_x\otimes det(\mathfrak{n}/s_{\mathfrak{n}}x)^*[-dim(\mathfrak{n}/s_{\mathfrak{n}}x)]$$

where $s_{\mathfrak{n}}x$ is the stabilizer in $\mathfrak{n}$ of $x\in G/B$.

Proof. We note that it is sufficient to prove this statement for $x=wB$ for $w\in W$, because for every $y\in G/B$, $\Gamma(G/B,\delta_y\otimes\mathcal{L}_\chi)$ is a twist of one of $\Gamma(G/B,\delta_{wB}\otimes\mathcal{L}_\chi)$ by an element of N.

Observe that $\Gamma(G/B,\delta_{wB}\otimes\mathcal{L}_\chi)$ is a simple $\mathfrak{g}$-module. So we can identify it with a semi-induced module of Voronov [24]. As a result[19] we obtain a description of $\Gamma(G/B,\delta_{wB}\otimes\mathcal{L}_\chi)$ as $\mathrm{Ind}_{\mathfrak{n}\cap\mathfrak{n}_w}^{\mathfrak{n}}\mathrm{Coind}_0^{\mathfrak{n}\cap\mathfrak{n}_w}\mathcal{L}_\chi|_{wB}$ as an $\mathfrak{n}$-module, where $\mathfrak{n}_w=w\mathfrak{n}w^{-1}$. The Lemma is then a consequence of the following isomorphisms that are versions of Shapiro Lemma:

$$\begin{aligned}H^\bullet(0,\mathcal{L}_\chi|_{wB}\otimes\det(\mathfrak{n}/\mathfrak{n}\cap\mathfrak{n}_w)^*)&\xleftarrow{\sim}H^\bullet(\mathfrak{n}\cap\mathfrak{n}_w,\mathrm{Coind}_0^{\mathfrak{n}\cap\mathfrak{n}_w}\mathcal{L}_\chi|_{wB}\otimes\det(\mathfrak{n}/\mathfrak{n}\cap\mathfrak{n}_w)^*)\\&\xrightarrow{\sim}H^\bullet(\mathfrak{n},\mathrm{Ind}_{\mathfrak{n}\cap\mathfrak{n}_w}^{\mathfrak{n}}\mathrm{Coind}_0^{\mathfrak{n}\cap\mathfrak{n}_w}\mathcal{L}_\chi|_{wB})[\dim(\mathfrak{n}/s_{\mathfrak{n}}x)],\end{aligned}$$

where the second isomorphism is Lemma 2.2. Note the use of triviality of the $\det(\mathfrak{n}/\mathfrak{n}\cap\mathfrak{n}_w)^*$ as an $\mathfrak{n}\cap\mathfrak{n}_w$-module.

□

[19] Alternatively, we can obtain the same result by transferring the D-module from X_w to X_{w_0} as in the proof of Theorem 2.1.

The Corollary below is a consequence of the identification of homology and cohomology explained in the proof of Lemma 2.2.

Corollary 2.4. *With the assumptions of Lemma 2.3,*

$$H_\bullet(\mathfrak{n}, \Gamma(G/B, \delta_x \otimes \mathcal{L}_\chi)) \cong \mathcal{L}_\chi|_x \otimes det(s_\mathfrak{n} x)[dim(s_\mathfrak{n} x)].$$

Remark. We follow the convention that dictates that the Lie algebra homology is placed in negative degrees. More precisely, $H_{-i}(\mathfrak{n}, M)$ is a subquotient of $\wedge^i \mathfrak{n} \otimes M$.

Theorem 2.5. *Let M be a right D-module on G/B, and $\chi - 2\rho$ be dominant regular, $n = dim(\mathfrak{n})$, then as $\mathfrak{h}$-modules*

$$H_\bullet(\mathfrak{n}, \Gamma(G/B, M \otimes \mathcal{L}_\chi)) \cong \bigoplus_{w \in W} H^\bullet_{DR}(X_w, i^!_w M) \otimes \mathbb{C}_{-w\cdot(\chi - 2\rho)}[n - \ell(w)].$$

Remark. We follow the convention that dictates that the De Rham cohomology is placed in both positive and negative degrees. More precisely, the left exact functor Γ is applied to the complex $\wedge^{-i}\mathcal{T}_X \otimes M$ that is confined to the negative degrees.

Proof. As in the proof of Theorem 2.1 we may reduce to a D-module of the form $i_{w*}M$ for some $w \in W$ and M a D-module on X_w. The action of N on X_w yields the following short exact sequence:

$$\mathcal{S}tab_w \xrightarrow{\alpha} \mathcal{O}_{X_w} \otimes \mathfrak{n} \xrightarrow{\beta} \mathcal{T}_{X_w}$$

where $\mathcal{S}tab_w$ is the kernel of the action map β. Choose a section s of β, define

$$\psi : \bigwedge^i \mathcal{T}_{X_w} \otimes \det(\mathcal{S}tab_w) \to \bigwedge^{i+n-\ell(w)} \mathfrak{n} \otimes \mathcal{O}_{X_w}$$

by $\omega \otimes v \mapsto s(\omega)v$, note that ψ does not depend on the choice of s. Then ψ extends to

$$\widetilde{\psi} : i_{w\cdot}(M \otimes \mathcal{L}_\chi|_{X_w} \otimes \bigwedge^\bullet \mathcal{T}_{X_w} \otimes \det(\mathcal{S}tab_w))[n - \ell(w)] \to i_{w*}M \otimes \mathcal{L}_\chi \otimes \bigwedge^\bullet \mathfrak{n}$$

where $\widetilde{\psi}$ is a morphism of complexes of sheaves on G/B that we intend to show is actually a quasi-isomorphism (after passing to $R\Gamma$ it yields the isomorphism of the Theorem).

Since the N-action trivializes both $\det(\mathcal{S}tab_w)$ and $\mathcal{L}_\chi|_{X_w}$ they contribute only a twist by a $\mathfrak{h}$-character and we have a map on the cohomologies:

$$H^\bullet_{DR}(X_w, M) \otimes \mathbb{C}_{-w\cdot(\chi-2\rho)}[n - \ell(w)] \to H_\bullet(\mathfrak{n}, \Gamma(G/B, i_{w*}M \otimes \mathcal{L}_\chi)).$$

Since M has a finite resolution by finite sums of $\mathcal{D}_{X_w}$ and their direct summands, we may assume that $M = \mathcal{D}_{X_w}$. In this case both sides are finite dimensional vector bundles over X_w and the map is a morphism of $\mathcal{O}_{X_w}$-modules. Over $x \in X_w$, the map becomes

$$H^\bullet_{DR}(X_w, \delta_x) \otimes \mathbb{C}_{-w\cdot(\chi-2\rho)}[n - \ell(w)] \to H_\bullet(\mathfrak{n}, \Gamma(G/B, i_{w*}\delta_x \otimes \mathcal{L}_\chi)),$$

which is an isomorphism by Corollary 2.4. This completes the proof. $\square$

The statement of Theorem 2.5 is made in terms of Lie algebra homology because the proof to us seemed most natural in that case (it avoids relative determinants for now), however it can be easily reformulated in terms of cohomology, namely

$$H^\bullet(\mathfrak{n}, \Gamma(G/B, M \otimes \mathcal{L}_\chi)) \cong \bigoplus_{w \in W} H^\bullet_{DR}(X_w, i^!_w M) \otimes \mathbb{C}_{w\cdot(-\chi)}[-\ell(w)],$$

compare this with the remark following Theorem 2.9.

Remark. It was pointed out to us by A. Beilinson that Theorem 2.1 (and thus Theorem 2.5) can be obtained as a consequence of the Beilinson-Bernstein localization theorem. The disadvantage of course is that while the proof given below is very simple, it uses both the center of $U\mathfrak{g}$, and the localization theorem of Beilinson-Bernstein; neither is present in the affine case.

Proof. Observe that

$$\begin{aligned} H^\bullet(\mathfrak{n}, \Gamma(G/B, M \otimes \mathcal{L}_\chi)) &\cong \mathrm{Ext}^\bullet_{U\mathfrak{n}}(\mathbb{C}, \Gamma(G/B, M \otimes \mathcal{L}_\chi)) \\ &\cong \mathrm{Ext}^\bullet_{U\mathfrak{g}}(U\mathfrak{g} \otimes_{U\mathfrak{n}} \mathbb{C}, \Gamma(G/B, M \otimes \mathcal{L}_\chi)). \end{aligned}$$

Since $\Gamma(G/B, M \otimes \mathcal{L}_\chi)$ is a module obtained from a twisted D-module, the center $\mathcal{Z}(U\mathfrak{g})$ of $U\mathfrak{g}$ acts on it via $\phi(-w_0(\chi))$, where $\phi : \mathfrak{h}^* \twoheadrightarrow \mathrm{Spec}\mathcal{Z}(U\mathfrak{g})$ is the Harish-Chandra map. Furthermore it acts in the same way on the Verma modules $\{V(w\cdot(-w_0(\chi)))|w \in W\} = \{V(w\cdot(-2\rho-\chi))|w \in W\}$. Note that $U\mathfrak{g} \otimes_{U\mathfrak{n}} \mathbb{C}$ on the other hand is a superposition of all Verma modules, which as a sheaf on $\mathrm{Spec}\mathcal{Z}(U\mathfrak{g})$ is locally free near $\phi(-w_0(\chi))$ as ϕ is étale there.

Let $\mathfrak{m}_\chi$ be the maximal ideal in $\mathcal{Z}(U\mathfrak{g})$ corresponding to $\phi(-w_0(\chi))$. Let $F_\bullet$ be the forgetful functor from the category of $U\mathfrak{g}/\mathfrak{m}_\chi$-modules to the category of $U\mathfrak{g}$-modules. It admits an obvious left adjoint F^*, namely the restriction to $\phi(-w_0(\chi)) \in \mathrm{Spec}\mathcal{Z}(U\mathfrak{g})$. The following chain of isomorphisms, with the third being the Beilinson-Bernstein localization theorem, completes this proof.

$$
\begin{aligned}
&\mathrm{Ext}^\bullet_{U\mathfrak{g}}(U\mathfrak{g} \otimes_{U\mathfrak{n}} \mathbb{C},\ F_\bullet\Gamma(G/B, M \otimes \mathcal{L}_\chi)) \\
&\qquad \cong \mathrm{Ext}^\bullet_{U\mathfrak{g}/\mathfrak{m}_\chi}(F^* U\mathfrak{g} \otimes_{U\mathfrak{n}} \mathbb{C}, \Gamma(G/B, M \otimes \mathcal{L}_\chi)) \\
&\qquad \cong \mathrm{Ext}^\bullet_{U\mathfrak{g}/\mathfrak{m}_\chi}(\bigoplus_{w\in W} V(w\cdot(-2\rho-\chi)), \Gamma(G/B, M \otimes \mathcal{L}_\chi)) \\
&\qquad \cong \bigoplus_{w\in W} \mathrm{Ext}^\bullet_{D_\chi-mod}(i_! \mathcal{O}_w \otimes \mathcal{L}_\chi, M \otimes \mathcal{L}_\chi) \otimes \mathbb{C}_{w\cdot(-2\rho-\chi)} \\
&\qquad \cong \bigoplus_{w\in W} \mathrm{Ext}^\bullet_{D-mod}(i_! \mathcal{O}_w, M) \otimes \mathbb{C}_{w\cdot(-2\rho-\chi)} \\
&\qquad \cong \bigoplus_{w\in W} \mathrm{Ext}^\bullet_{D_{X_w}-mod}(\mathcal{O}_w, i_w^! M) \otimes \mathbb{C}_{w\cdot(-2\rho-\chi)} \\
&\qquad \cong \bigoplus_{w\in W} H^\bullet_{DR}(X_w, i_w^! M) \otimes \mathbb{C}_{w\cdot(-2\rho-\chi)}
\end{aligned}
$$

□

The geometric computation, in this section, of the cohomology $H^\bullet(\mathfrak{n}, V)$, where V is a $\mathfrak{g}$-module that comes from a D-module on G/B, can be viewed as a recipe for reconstructing the original geometric object, namely the D-module, from the algebraic data of V and its cohomology. Informally, $H^\bullet(\mathfrak{n}, V)$ is computed from the de Rham cohomology of the restriction of the D-module to the N-orbits. By considering V^g, i.e. g-twists of V via the adjoint action, as $g \in G$ varies, we can reconstruct the D-module. In fact twisting V by g is equivalent, on the D-module side, to the pullback along the action of g on G/B. Thus $H^\bullet(\mathfrak{n}, V^g)$ (with varying g) contains the data of the de Rham cohomology of the restriction of the D-module to the gNg^{-1}-orbits. This is sufficient to recover the D-module; it is very natural in view of the fact that one of the orbits is a point, and varying g allows the freedom of making this point, any point on G/B.

Let us be more precise in the following case that illustrates the general situation. Suppose that V is a $\mathfrak{g}$-module with the trivial central character. By the Beilinson-Bernstein localization theorem we know that it comes from a D-module, i.e. we have $V = \Gamma(G/B, M)$ for some D-module M; let us recover it. We have the usual short exact sequence

$$0 \to \underline{\mathfrak{b}} \to \mathcal{O}_{G/B} \otimes \mathfrak{g} \to \tau_{G/B} \to 0$$

where $\mathcal{O}_{G/B} \otimes \mathfrak{g}$ is the action Lie algebroid on G/B, the surjection onto the vector fields is the anchor map, and $\underline{\mathfrak{b}}$ is the kernel of the anchor map; let $\underline{\mathfrak{n}} = [\underline{\mathfrak{b}}, \underline{\mathfrak{b}}]$ and $\underline{\mathfrak{b}}/\underline{\mathfrak{n}} = \mathcal{O}_{G/B} \otimes \mathfrak{h}$. Then $H^{\ell(w_0)}(\underline{\mathfrak{n}}, \mathcal{L}_{-2\rho} \otimes V)^{\mathfrak{h}}$ is a D-module, and it follows from Theorem 2.1 that

$$H^{\ell(w_0)}(\underline{\mathfrak{n}}, \mathcal{L}_{-2\rho} \otimes V)^{\mathfrak{h}} = M$$

and this is essentially the localization of V. Ultimately one wishes to do the same in the affine setting, the problem is that there is no point orbit, however in principle it should still be possible to recover the D-module from its de Rham data.

2.2 The affine setting.

Let us now deal with the affine setting, namely we turn our attention to (right) D-modules on $\mathcal{F}\ell$. Since we will be working with the affine Grassmannian $\mathcal{G}r$ and the affine flags $\mathcal{F}\ell$ extensively in what follows, we say a few words about them at this point. Recall that $\mathcal{K}$ is the ring of Laurent series $\mathbb{C}((t))$ and we denote by $G(\mathcal{K})$ the group parameterizing maps of the formal punctured disk $D^\times$ to G. We have some natural subgroups: $G(\mathcal{O})$ which parameterizes maps of the formal disk D to G and I which is a subgroup of the latter that consists only of those maps whose center lands in $B \subset G$. Then, roughly speaking, $\mathcal{F}\ell = G(\mathcal{K})/I$ and its quotient $\mathcal{G}r$ is $G(\mathcal{K})/G(\mathcal{O})$. This description is sufficient for following our geometric arguments, however for the reader interested in the foundations we point out that both can be given the structure of an ind-scheme of ind-finite type. Furthermore, $\mathcal{G}r$ possesses factorization space structure and $\mathcal{F}\ell$ is a factorization module space over $\mathcal{G}r$. We refer the reader to [18, 22] for the precise formulations.

In the affine setting we have a choice in generalizing the finite dimensional situation. We can consider the relationship between Iwahori orbits and Lie algebra cohomology, or alternatively semi-infinite orbits and semi-infinite cohomology. The latter is better suited to our purposes and so we focus on it, briefly mentioning the former in the remark at the end of the section.

We begin with some preliminary Lemmas establishing the shifts and twists that will appear later in the semi-infinite cohomology computations. The reader is strongly encouraged to refer to Sec. 1.1 when following the discussion below. Let $w \in W_{\text{aff}}$, $w = \lambda_w \bar{w}$ (with $\lambda_w \in \Gamma$ and $\bar{w} \in W$), set $\mathfrak{i}_w = w\mathfrak{i}w^{-1}$ then $\mathfrak{n}(\mathcal{K}) \cap \mathfrak{i}_w$ is a semi-infinite subspace of $\mathfrak{n}(\mathcal{K})$. We are interested in computing the character of the relative determinant $\det = \det(\mathfrak{n}(\mathcal{K}) \cap \mathfrak{i}_w, \mathfrak{n}(\mathcal{O}))$ as a $\mathfrak{h}$-module, as well as the relative dimension $\dim = \dim(\mathfrak{n}(\mathcal{K}) \cap \mathfrak{i}_w, \mathfrak{n}(\mathcal{O}))$. Recall that for a pair of semi-infinite subspaces U and V,

$$\det(U,V) = \det(U/(U \cap V)) \otimes \det(V/(U \cap V))^*$$

which makes sense since both $U/(U \cap V)$ and $V/(U \cap V)$ are finite dimensional, and similarly

$$\dim(U,V) = \dim(U/(U \cap V)) - \dim(V/(U \cap V))$$

so that the relative dimension is an integer that need not be non-negative.

Lemma 2.6. *We have det* $\cong \mathbb{C}_{\bar{w}\cdot 0+\kappa_c(\lambda_w)}$ *and dim* $= -2ht\lambda_w - \ell(\bar{w})$.

Proof. Observe that

$$\begin{aligned}\det(\mathfrak{n}(\mathcal{K})\cap w\mathfrak{g}(\mathcal{O})w^{-1}, \mathfrak{n}(\mathcal{O}))& \\ &= \det((\mathfrak{n}(\mathcal{K})\cap \mathfrak{i}_w)\oplus(\mathfrak{n}(\mathcal{K})\cap w\mathfrak{n}^- w^{-1}), \mathfrak{n}(\mathcal{O})) \\ &\cong \det\otimes\det(\mathfrak{n}(\mathcal{K})\cap w\mathfrak{n}^- w^{-1}) \\ &= \det\otimes\det(\mathfrak{n}\cap \bar{w}\mathfrak{n}^-\bar{w}^{-1}) \\ &\cong \det\otimes\mathbb{C}_{-\bar{w}\cdot 0}.\end{aligned}$$

While at the same time

$$\begin{aligned}\det(\mathfrak{n}(\mathcal{K})\cap w\mathfrak{g}(\mathcal{O})w^{-1}, \mathfrak{n}(\mathcal{O}))& \\ &= \det(\mathfrak{n}(\mathcal{K})\cap \lambda_w\mathfrak{g}(\mathcal{O})\lambda_w^{-1}, \mathfrak{n}(\mathcal{O})) \\ &= \det(\lambda_w\mathfrak{n}(\mathcal{O})\lambda_w^{-1}, \mathfrak{n}(\mathcal{O})) \\ &\cong \bigotimes_{\alpha>0}\mathbb{C}_{-\alpha(\lambda_w)\alpha} \\ &= \mathbb{C}_{\kappa_c(\lambda_w)}.\end{aligned}$$

So that $\det\cong\mathbb{C}_{\bar{w}\cdot 0+\kappa_c(\lambda_w)}$. Identically, $\dim = -2\mathrm{ht}\lambda_w - \ell(\bar{w})$. □

At this point we require an analogue of Lemma 2.3. Recall that $\mathfrak{n}(\mathcal{K})_\dagger$ and $\mathfrak{n}(\mathcal{O})_\dagger$ denote $\mathfrak{n}(\mathcal{K})\oplus t\mathfrak{h}(\mathcal{O})$ and $\mathfrak{n}(\mathcal{O})\oplus t\mathfrak{h}(\mathcal{O})$ respectively. Let us review some basics of the semi-infinite cohomology of $\mathfrak{n}(\mathcal{K})_\dagger$ (it is somewhat simpler than the general case). The complex $\bigwedge^{\infty/2+\bullet}$ is obtained by taking the quotient of the Clifford algebra $C := C(\mathfrak{n}(\mathcal{K})_\dagger\oplus\mathfrak{n}(\mathcal{K})_\dagger^*)$ by the left ideal generated by the standard semi-infinite isotropic subspace $L\subset\mathfrak{n}(\mathcal{K})_\dagger\oplus\mathfrak{n}(\mathcal{K})_\dagger^*$. Thus $\bigwedge^{\infty/2+\bullet} = C/(C\cdot L)$, where $L := \mathfrak{n}(\mathcal{O})_\dagger\oplus(\mathfrak{n}(\mathcal{K})_\dagger/\mathfrak{n}(\mathcal{O})_\dagger)^*$; note that $\bigwedge^{\infty/2+\bullet}$ is a $\mathfrak{h}$-module. Let $|0\rangle$ denote the image of 1 in the quotient. The differential is written analogously with the usual cohomology differential, namely it is $-\frac{1}{2}f^\alpha_{\beta\gamma} : c^\beta c^\gamma c_\alpha :$, where $f^\alpha_{\beta\gamma}$ are the structure constants of the Lie algebra $\mathfrak{n}(\mathcal{K})_\dagger$ with respect to the usual basis ($\{c_\alpha\}$ is the basis of $\mathfrak{n}(\mathcal{K})_\dagger$, $\{c^\alpha\}$ the dual basis of $\mathfrak{n}(\mathcal{K})_\dagger^*$). Note that because $f^\alpha_{\beta\gamma}$ vanish whenever any two of the indices coincide, the normal ordering $: c^\beta c^\gamma c_\alpha :$ does not change the differential. The situation modifies readily to the case of coefficients in an $\mathfrak{n}(\mathcal{K})_\dagger$-module M. Namely, the differential in this case is given by

$$c_\alpha\otimes c^\alpha - \frac{1}{2}f^\alpha_{\beta\gamma} : c^\beta c^\gamma c_\alpha : \in U(\mathfrak{n}(\mathcal{K})_\dagger)\otimes C$$

acting on $M\otimes\bigwedge^{\infty/2+\bullet}$.

The complex described above gives the standard semi-infinite cohomology of $\mathfrak{n}(\mathcal{K})_\dagger$, however one may choose a different semi-infinite split of $\mathfrak{n}(\mathcal{K})_\dagger$, i.e., a different isotropic subspace of $\mathfrak{n}(\mathcal{K})_\dagger\oplus\mathfrak{n}(\mathcal{K})_\dagger^*$ and obtain a complex that way (as we will see below there is not much difference, this is analogous to the similarity between homology and cohomology in the finite case, see the proof of Lemma 2.2).

We will be particularly interested in the following. Let $w \in W_{\mathrm{aff}}$, recall that $\mathfrak{i}_w = w \mathfrak{i} w^{-1}$. Then $L_w := (\mathfrak{n}(\mathcal{K})_\dagger \cap \mathfrak{i}_w) \oplus (\mathfrak{n}(\mathcal{K})_\dagger / (\mathfrak{n}(\mathcal{K})_\dagger \cap \mathfrak{i}_w))^*$ is an isotropic semi-infinite subspace of $\mathfrak{n}(\mathcal{K})_\dagger \oplus \mathfrak{n}(\mathcal{K})_\dagger^*$ (taking w to be identity in W_{aff} we recover the standard case). We may now consider the complex $\bigwedge_w^{\infty/2+\bullet}$ formed by taking the quotient of the Clifford algebra C by the left ideal generated by L_w, let $|0\rangle_w$ denote the image of 1 in the quotient. The differential is still given by the same formula (it is an element of $C(\mathfrak{n}(\mathcal{K})_\dagger \oplus \mathfrak{n}(\mathcal{K})_\dagger^*)$ and so acts on any module over the Clifford algebra), except now the normal ordering is taken with respect to a different semi-infinite split. However, as is explained above, for the special case of $\mathfrak{n}(\mathcal{K})_\dagger$, this does not change the differential.

Any choice of a non-zero element v_w in the line $\det(\mathfrak{n}(\mathcal{K})_\dagger \cap \mathfrak{i}_w, \mathfrak{n}(\mathcal{O})_\dagger)$ yields a map of C-modules as follows

$$\begin{aligned} \phi_{v_w} : \bigwedge\nolimits_w^{\infty/2+\bullet} &\longrightarrow \bigwedge\nolimits^{\infty/2+\bullet}[-\dim(\mathfrak{n}(\mathcal{K})_\dagger \cap \mathfrak{i}_w, \mathfrak{n}(\mathcal{O})_\dagger)] \\ |0\rangle_w &\mapsto v_w\,|0\rangle \end{aligned}$$

where the expression $v_w\,|0\rangle$ is well defined since $v_w \in \wedge^\bullet(\mathfrak{n}(\mathcal{K})_\dagger/\mathfrak{n}(\mathcal{O})_\dagger) \otimes \wedge^\bullet \mathfrak{n}(\mathcal{O})_\dagger^*$. This map is an isomorphism of C-modules, and since the differential for both modules is given by the same element of C, it is an isomorphism of complexes. Note that ϕ_{v_w} shifts the grading as indicated and twists the $\mathfrak{h}$-action. Given a $\hat{\mathfrak{g}}_\kappa$-module M, we denote by $H_w^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K})_\dagger, M)$ the cohomology of the complex $M \otimes \bigwedge_w^{\infty/2+\bullet}$ and omit the subscript w when it is the identity. Observe that $H_w^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K})_\dagger, M)$ is a $\mathfrak{h}$-module.

Lemma 2.7. *Let δ_x be the right D-module of delta functions at $x \in \mathcal{F}\ell$, $\chi - 2\rho$ dominant regular, and κ sufficiently negative then*

$$\begin{aligned} H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K})_\dagger, &\Gamma(\mathcal{F}\ell, \delta_x \otimes \mathcal{L}_{\chi+\kappa})) \\ &\cong \mathcal{L}_{\chi+\kappa}|_x \otimes det(s_{\mathfrak{n}(\mathcal{K})_\dagger} x, \mathfrak{n}(\mathcal{O})_\dagger)\,|0\rangle\,[dim(s_{\mathfrak{n}(\mathcal{K})_\dagger} x, \mathfrak{n}(\mathcal{O})_\dagger)] \end{aligned}$$

where $s_{\mathfrak{n}(\mathcal{K})_\dagger} x$ is the stabilizer in $\mathfrak{n}(\mathcal{K})_\dagger$ of $x \in \mathcal{F}\ell$.

Remark. The meaning of *sufficiently negative* in the statement of Lemma 2.7 and theorems below is as follows. We require first of all that $\Gamma(\mathcal{F}\ell, \delta_x \otimes \mathcal{L}_{\chi+\kappa})$ be irreducible as a $\hat{\mathfrak{g}}_\kappa$-module for any $x \in \mathcal{F}\ell$. This reduces to irreducibility of the Verma module $M_{-\chi,\kappa}$, which using [20] can be shown to be irreducible whenever

$$\kappa - \kappa_c \leq -(\chi - \rho, \theta)_0,$$

i.e., whenever $\kappa \leq -(\chi, \theta)_0 - 1$. We also need the exactness of the functor $\Gamma(\mathcal{F}\ell, - \otimes \mathcal{L}_{\chi+\kappa})$, however by [5] this holds for the case when $M_{-\chi,\kappa}$ is irreducible, so the above condition is sufficient for this as well. Another condition is needed to ensure the degeneration of the spectral sequence that allows us to consider only D-modules supported on a single orbit, i.e., we need that $w \cdot (-\chi) \neq w' \cdot (-\chi)$ in $\mathfrak{h}^*$ whenever

$w \neq w'$ in W_{aff}.[20] We point out that this is a requirement on the *orbit* of $-\chi$ under the dot W_{aff} action; we ask that the action have no stabilizer. We can describe, completely combinatorially, a sufficient condition for this to hold i.e., for every $w \in W$, w not identity, there is a root α of $\mathfrak{g}$ such that $\left(\alpha, \frac{\chi-\rho-w(\chi-\rho)}{\kappa-\kappa_c}\right)_0 \notin \mathbb{Z}$. If we require that

$$\kappa - \kappa_c < -2(\chi - \rho, \theta)_0,$$

then it suffices both for this and the previous two requirements.

Proof. We begin exactly as we did in the proof of Lemma 2.3. Reduce to the case of $x = wI$, and note that $\Gamma(\mathcal{F}\ell, \delta_{wI} \otimes \mathcal{L}_{\chi+\kappa})$ is a simple module (by the Remark above). Then, just as in the finite case, it can be identified with an appropriate semi-induced module of Voronov. Consequently (by the semi-infinite Shapiro Lemma of [24]) we see that

$$H_w^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K})_\dagger, \Gamma(\mathcal{F}\ell, \delta_{wI} \otimes \mathcal{L}_{\chi+\kappa})) \cong \mathcal{L}_{\chi+\kappa}|_{wI} \otimes \mathbb{C}\left|0\right\rangle_w .$$

Thus, according to the discussion preceding the present Lemma,

$$\begin{aligned} &H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K})_\dagger, \Gamma(\mathcal{F}\ell, \delta_{wI} \otimes \mathcal{L}_{\chi+\kappa})) \\ &\qquad \cong \mathcal{L}_{\chi+\kappa}|_{wI} \otimes \det(\mathfrak{n}(\mathcal{K})_\dagger \cap \mathfrak{i}_w, \mathfrak{n}(\mathcal{O})_\dagger) \left|0\right\rangle [\dim(\mathfrak{n}(\mathcal{K})_\dagger \cap \mathfrak{i}_w, \mathfrak{n}(\mathcal{O})_\dagger)]. \end{aligned}$$

This completes the proof.

□

We now prove the semi-infinite affine analogue of Theorem 2.5. Recall that

$$w \cdot \chi = \bar{w} \cdot \chi - (\kappa - \kappa_c)(\lambda_w)$$

is the affine dot action of W_{aff} on $\mathfrak{h}^*$ corresponding to the level $\kappa - \kappa_c$.

Proposition 2.8. *Let M be a D-module on $\mathcal{F}\ell$, $S_w \subset \mathcal{F}\ell$ the $N(\mathcal{K})$ orbit labeled by $w \in W_{aff}$, suppose $\chi - 2\rho$ is dominant regular, κ sufficiently negative, then as $\mathfrak{h}$-modules*

$$\begin{aligned} &H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K})_\dagger, \Gamma(\mathcal{F}\ell, M \otimes \mathcal{L}_{\chi+\kappa})) \\ &\qquad \cong \bigoplus_{w \in W_{aff}} H^\bullet_{DR}(S_w, i_w^! M) \otimes \mathbb{C}_{w\cdot(-\chi)}[-2ht(\lambda_w) - \ell(\bar{w})]. \end{aligned}$$

Proof. We begin by observing that we can reduce to the special case of $M = i_{w*}M_0$ similarly to the finite dimensional case (we use that κ is sufficiently negative here). Now let M be a D-module on S_w. Below we construct an explicit map from the De Rham to the semi-infinite cohomology.

[20] The remark following Theorem 2.9 can be used to show that this is also a necessary condition for the functor $\Gamma(\mathcal{F}\ell, - \otimes \mathcal{L}_{\chi+\kappa})$ to be an embedding.

Consider the short exact sequence of vector bundles on S_w arising from the action of $N(\mathcal{K})_\dagger$ on its orbit:

$$\mathcal{S}tab_w \xrightarrow{\alpha} \mathcal{O}_{S_w} \otimes \mathfrak{n}(\mathcal{K})_\dagger \xrightarrow{\beta} \mathcal{T}_{S_w}$$

and denote by $\mathcal{L}_{det}$ the relative determinant line bundle $\det(\mathcal{S}tab_w, \mathcal{O}_{S_w} \otimes \mathfrak{n}(\mathcal{O})_\dagger)$, so that we have a natural map

$$\psi : \bigwedge^\bullet \mathcal{T}_{S_w} \otimes \mathcal{L}_{\det}[\dim] \to \mathcal{O}_{S_w} \otimes \bigwedge^{\infty/2+\bullet} \mathfrak{n}(\mathcal{K})_\dagger$$

as in the proof of Theorem 2.5. This is known as the "fermions canceling the determinantal anomaly". Similarly, ψ extends to

$$\widetilde{\psi} : i_{w\cdot}(M \otimes \mathcal{L}_{\kappa+\chi}|_{S_w} \otimes \bigwedge^\bullet \mathcal{T}_{S_w} \otimes \mathcal{L}_{\det})[\dim] \to i_{w*}M \otimes \mathcal{L}_{\kappa+\chi} \otimes \bigwedge^{\infty/2+\bullet} \mathfrak{n}(\mathcal{K})_\dagger$$

that is a morphism of complexes of sheaves on $\mathcal{F}\ell$.

Note that $\mathcal{L}_{\det} \otimes \mathcal{L}_{\kappa+\chi}|_{S_w}$ is canonically trivialized by the $N(\mathcal{K})_\dagger$ action contributing only a twist by a $\mathfrak{h}$ character $(\mathcal{L}_{\det} \otimes \mathcal{L}_{\kappa+\chi}|_{S_w})|_{wI} \cong \mathbb{C}_{w\cdot(-\chi)}$. This is where we use Lemma 2.6, the only difference is the extra $t\mathfrak{h}(\mathcal{O})$ term that does not affect the computation. There is also a shift by dim that was already noted in the above, thus on the level of cohomology we have

$$\begin{aligned} &H^\bullet_{DR}(S_w, M) \otimes \mathbb{C}_{w\cdot(-\chi)}[-2\mathrm{ht}(\lambda_w) - \ell(\bar{w})] \\ &\qquad\qquad \longrightarrow H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K})_\dagger, \Gamma(\mathcal{F}\ell, i_{w*}M \otimes \mathcal{L}_{\chi+\kappa})). \end{aligned}$$

The map above commutes with direct limits, so it is sufficient to consider the case when M is coherent with finite dimensional support, so that $M = i_*M_0$, with $i : X \hookrightarrow S_w$ the inclusion of a smooth finite dimensional X that contains the support (such an X exists since S_w is smooth). Then M_0 has a finite resolution by finite sums of $\mathcal{D}_X$ and their direct summands, and so we may assume that $M_0 = \mathcal{D}_X$. In that case both sides of the above can be considered $\mathcal{O}_X$-modules (locally free), and the map becomes an $\mathcal{O}_X$ morphism. It is thus sufficient to check that it is an isomorphism on every fiber. This reduces to checking the statement for $M_0 = \delta_x$ with $x \in X$, and that is the content of Lemma 2.7.

□

One is actually interested in the BRST reduction, which has the advantage of producing a vertex algebra if we begin with one. The following addresses that issue.

Theorem 2.9. *Let M be a D-module on $\mathcal{F}\ell$, S_w the $N(\mathcal{K})$ orbit labeled by $w \in W_{aff}$, π_α the irreducible $\hat{\mathfrak{h}}_{\kappa-\kappa_c}$-module of highest weight α, suppose $\chi - 2\rho$*

is dominant regular, κ sufficiently negative, then as $\hat{\mathfrak{h}}_{\kappa-\kappa_c}$-modules[21]

$$\begin{aligned} &H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), \Gamma(\mathcal{F}\ell, M\otimes\mathcal{L}_{\chi+\kappa})) \\ &\qquad \cong \bigoplus_{w\in W_{\mathit{aff}}} H^{\bullet}_{DR}(S_w, i_w^! M)\otimes \pi_{w\cdot(-\chi)}[-2ht(\lambda_w)-\ell(\bar{w})]. \end{aligned}$$

Proof. An analogue of the Kashiwara theorem states that the category of $\hat{\mathfrak{h}}_{\kappa-\kappa_c}$-modules with locally nilpotent $t\mathfrak{h}(\mathcal{O})$ action is equivalent to the category of $\mathfrak{h}$-modules, with $t\mathfrak{h}(\mathcal{O})$ invariants in one direction and induction in the other giving the equivalence[22]. Thus it suffices to show that $t\mathfrak{h}(\mathcal{O})$ acts locally nilpotently on $H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), \Gamma(\mathcal{F}\ell, M\otimes\mathcal{L}_{\chi+\kappa}))$, or in light of the above we may assume that $M = i_{w*}i_w^! M$ for a fixed $w\in W_{\mathrm{aff}}$. As before we may reduce this to the case $M = i_{w*}i_*\mathcal{D}_X$, making sure that $wI\in X$. So that as a $t\mathfrak{h}(\mathcal{O})\otimes\mathcal{O}_X$-module

$$H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), \Gamma(\mathcal{F}\ell, i_{w*}i_*\mathcal{D}_X\otimes\mathcal{L}_{\chi+\kappa})) \cong \mathcal{O}_X\otimes H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), \Gamma(\mathcal{F}\ell, \delta_{wI}\otimes\mathcal{L}_{\chi+\kappa})).$$

This is sufficient since the $t\mathfrak{h}(\mathcal{O})$ action on both $\Gamma(\mathcal{F}\ell, \delta_{wI}\otimes\mathcal{L}_{\chi+\kappa})$ and $\bigwedge^{\infty/2+\bullet}\mathfrak{n}(\mathcal{K})$ is locally nilpotent.

□

Remark. Considering instead Iwahori orbits and the usual cohomology one has the formula:

$$H^{\bullet}(\mathfrak{i}^+, \Gamma(\mathcal{F}\ell, M\otimes\mathcal{L}_{\kappa+\chi})) \cong \bigoplus_{w\in W_{\mathrm{aff}}} H^{\bullet}_{DR}(X_w, i_w^! M)\otimes\mathbb{C}_{w\cdot(-\chi)}[-\ell(w)]$$

as $\mathfrak{h}$-modules. This can be shown using averaging (thus reducing the general problem to the case of constant D-modules on orbits that correspond to co-Verma modules). A proof of Proposition 2.8 can then be extracted from the consideration of the above formula for an appropriate sequence of Iwahori conjugates. This is the approach suggested by A. Beilinson and D. Gaitsgory and followed in [23].

Remark. If M is a right D-module on G/B and $i : G/B \hookrightarrow \mathcal{F}\ell$ is the inclusion of the fiber of $p : \mathcal{F}\ell \twoheadrightarrow \mathcal{G}r$ over $G(\mathcal{O})$, then the natural map

$$H^{\bullet}(\mathfrak{n}, \Gamma(G/B, M\otimes\mathcal{L}_{\chi})) \longrightarrow H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), \Gamma(\mathcal{F}\ell, i_*M\otimes\mathcal{L}_{\chi+\kappa}))$$

is an isomorphism onto the highest weights.

[21] The $\hat{\mathfrak{h}}_{\kappa-\kappa_c}$-module π_0 is known as the Heisenberg vertex algebra, and its representation theory is the same as that of $\hat{\mathfrak{h}}_{\kappa-\kappa_c}$. Thus Theorem 2.9 is equivalently viewed as describing the BRST reduction as a π_0-module.

[22] This statement, without refering to it as Kashiwara theorem, is explained in [10].

3 The BRST reduction.

Let A be a $\check{G}$-module, then by the geometric Satake isomorphism [19, 21] there is a $G(\mathcal{O})$-equivariant D-module $\mathcal{A}$ on $\mathcal{G}r$ such that $H^{\bullet}_{DR}(\mathcal{G}r, \mathcal{A}) = A$ (disregarding the grading, in fact the cohomology is rarely concentrated in degree 0). Let us compute the BRST reduction of $\Gamma(\mathcal{G}r, \mathcal{A} \otimes \mathcal{L}_\kappa)$. The tools are Theorem 2.9 and the Mirković-Vilonen theorem [21, 22].

Proposition 3.1. *Let $A(\lambda)$ denote the λ weight space of A, then as $\hat{\mathfrak{h}}_{\kappa-\kappa_c}$-modules*

$$\begin{aligned} &H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), \Gamma(\mathcal{G}r, \mathcal{A} \otimes \mathcal{L}_\kappa)) \\ &\qquad \cong \bigoplus_{\substack{\lambda\in\Gamma \\ w\in W}} A(\lambda) \otimes \pi_{w\cdot 0-(\kappa-\kappa_c)\lambda}[-\ell(w)]. \end{aligned}$$

Proof. The notation comes from the diagram below.

$$\begin{tikzcd} \mathcal{F}\ell \arrow[d, "p"'] & S_w \arrow[l, hook', "i_w"'] \arrow[d, "\tilde{p}"] \\ \mathcal{G}r & S_{\lambda_w} \arrow[l, hook', "i_{\lambda_w}"] \end{tikzcd}$$

We begin by observing that $\Gamma(\mathcal{G}r, \mathcal{A} \otimes \mathcal{L}_\kappa) = \Gamma(\mathcal{F}\ell, p^*\mathcal{A} \otimes \mathcal{L}_{\kappa+2\rho})$, since the fibres of p are (non-canonically) G/B, i.e., compact; the pullback is of right D-modules and so we need the factor $\mathcal{L}_{2\rho}$ to make sure that $p^*\mathcal{A} \otimes \mathcal{L}_{\kappa+2\rho}$, when restricted to the fibres of p, is just $\mathcal{O}_{G/B}$.

To apply Theorem 2.9 we will need $H^{\bullet}_{DR}(S_w, i^!_w p^*\mathcal{A})$, while the Mirković-Vilonen theorem tells us that $H^{\bullet}_{DR}(S_\lambda, i^!_\lambda \mathcal{A}) = A(\lambda)[2\mathrm{ht}\lambda]$. Note that since p is smooth with fiber G/B, and $\tilde{p}$ is smooth with fiber $X_{\bar{w}}$, we observe that $i^!_w p^*\mathcal{A} \cong \tilde{p}^* i^!_{\lambda_w}\mathcal{A}[-\ell(w_0) + \ell(\bar{w})]$. Thus

$$H^{\bullet}_{DR}(S_w, i^!_w p^*\mathcal{A}) \cong H^{\bullet}_{DR}(S_{\lambda_w}, i^!_{\lambda_w}\mathcal{A})[-\ell(w_0) + 2\ell(\bar{w})].$$

This together with re-indexing, and setting $w = ww_0$ yields the result.

∎

Remark. The method of the proof above can be used to give a version of Theorem 2.9 for the affine Grassmannian $\mathcal{G}r$ with $N(\mathcal{K})$ orbits S_λ indexed by $\lambda \in \Gamma$. Namely, as $\hat{\mathfrak{h}}_{\kappa-\kappa_c}$-modules

$$\begin{aligned} &H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), \Gamma(\mathcal{G}r, M \otimes \mathcal{L}_\kappa)) \\ &\qquad \cong \bigoplus_{\substack{\lambda\in\Gamma \\ w\in W}} H^{\bullet}_{DR}(S_\lambda, i^!_\lambda M) \otimes \pi_{w\cdot 0-(\kappa-\kappa_c)\lambda}[-2\mathrm{ht}\lambda - \ell(w)]. \end{aligned}$$

Let $A = \mathcal{O}_{\check{G}}$, and call the resulting $\hat{\mathfrak{g}}_\kappa$-module $S_\kappa(G)$. Observe that $S_\kappa(G)$ is actually a $\check{G} \times \hat{\mathfrak{g}}_\kappa$-module due to the other action of $\check{G}$ on $\mathcal{O}_{\check{G}}$. Define the following $\check{H} \times \hat{\mathfrak{h}}_{\kappa-\kappa_c}$-module

$$\mathcal{V}^\bullet = \bigoplus_{\substack{\lambda\in\Gamma \\ w\in W}} \mathbb{C}_{-\lambda} \otimes \pi_{w\cdot 0-(\kappa-\kappa_c)\lambda}[-\ell(w)].$$

Corollary 3.2. *As a $\check{G} \times \hat{\mathfrak{h}}_{\kappa-\kappa_c}$-module*

$$H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), S_\kappa(G)) \cong \Gamma(\check{G}/\check{H}, \check{G} \times_{\check{H}} \mathcal{V}^\bullet).$$

Proof. By Proposition 3.1

$$H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), S_\kappa(G)) \cong \bigoplus_{\substack{\lambda\in\Gamma \\ w\in W}} \bigoplus_{\chi\in\Gamma^+} V_\chi^* \otimes V_\chi(\lambda) \otimes \pi_{w\cdot 0-(\kappa-\kappa_c)\lambda}[-\ell(w)]$$

and $\bigoplus_{\chi\in\Gamma^+} V_\chi^* \otimes V_\chi(\lambda)$ naturally identifies with $\Gamma(\check{G}/\check{H}, \check{G} \times_{\check{H}} \mathbb{C}_{-\lambda})$. ☐

3.1 The chiral structure.

If we assume that A above is also a unital $\check{G}$-equivariant commutative algebra, then by formal considerations we see that $A_\kappa(\mathfrak{g}) = \Gamma(\mathcal{G}r_G, \mathcal{A}_G \otimes \mathcal{L}_\kappa)$[23] is a vertex algebra, with a vertex subalgebra $V_\kappa(\mathfrak{g})$ coming from the unit. See Sec. 4.1 for more details. Note that if $A = \Gamma(X, \mathcal{B})$ where $\mathcal{B}$ is a bundle of $\check{G}$-equivariant commutative algebras then $A_\kappa(\mathfrak{g})$ also fibers over X and the fibres are $A_\kappa(\mathfrak{g})_x = (\mathcal{B}_x)_\kappa(\mathfrak{g})$.

For our purposes, it is also useful to consider A as a $\check{H}$-module, and via a similar procedure we obtain another vertex algebra $A_\kappa(\mathfrak{h}) = \Gamma(\mathcal{G}r_H, \mathcal{A}_H \otimes \mathcal{L}_\kappa)$.

Remark. Starting with a $\check{H}$-algebra $\Gamma(\check{H}, \mathcal{O}_{\check{H}})$ and proceeding as above we get the lattice Heisenberg vertex algebra. When $A = \mathcal{O}_{\check{G}}$, then $A_\kappa(\mathfrak{g}) = S_\kappa(G)$, known as the chiral Hecke algebra (Sec. 4.1).

Thus the BRST reduction of $A_\kappa(\mathfrak{g})$ is not only a $\hat{\mathfrak{h}}_{\kappa-\kappa_c}$-module, but also a vertex algebra and below we describe the vertex algebra structure on its subalgebra $H^{\infty/2+0}(\mathfrak{n}(\mathcal{K}), A_\kappa(\mathfrak{g}))$. First we need a Lemma. Let V be a finite dimensional vector space, $\mathcal{L}_{\det}$ the canonical determinant factorization line bundle on $\mathcal{G}r_{GL(V)}$, and $\mathcal{C}\ell^\bullet$ the constant bundle with fiber $\bigwedge^\bullet_V$.

In what follows we briefly switch to the language of factorization algebras as the constructions involved are performed most naturally in that setting. The languages of vertex algebras, chiral algebras and factorization algebras can be used essentially interchangeably and [10] is an excellent dictionary. In the proof below we use effective divisors on a curve X instead of the points in X as the reader may be used to. We point out that this is basically the same thing as X is one

[23] These are just our $\mathcal{G}r$ and $\mathcal{A}$ from before. We will soon need to distinguish between different Grassmanians.

dimensional (thus effective divisors are just points with multiplicities[24]). However, effective divisors make sense in families and this is necessary for a proper definition of factorization structure (which is essentially a description of what happens when points collide).

Lemma 3.3. *$\mathcal{C}\ell^\bullet$ has factorization structure and the canonical map*

$$\mathcal{L}_{det} \longrightarrow \mathcal{C}\ell^\bullet$$

is compatible with factorization structures.

Remark. It was communicated to us by A. Beilinson that the Lemma is a special case, with $G = GL_n$, of a very general situation which makes sense for an arbitrary reductive group G. Namely, consider the vacuum integrable representation V of $G(\mathcal{K})^\sim$ of level κ. This is naturally a vertex algebra (a quotient of the usual Kac-Moody vertex algebra). V can be realized as the dual vector space to the space of sections of a certain positive line bundle on $\mathcal{G}r$. This line bundle admits a canonical factorization structure, and the dual line bundle $\mathcal{L}$ embeds naturally into $V \otimes \mathcal{O}_{\mathcal{G}r}$ in a way compatible with the factorization structures.

Proof. Since $\bigwedge_V^\bullet$ is a vertex algebra[25], $\mathcal{C}\ell^\bullet$ has factorization structure. From this description of the structure one can not see directly why the natural map above is compatible with it. There is a construction, due to A. Beilinson, that is very similar on the level of vector spaces to the one in [10], but which very naturally (i.e., without formulas) produces a factorization structure. Almost tautologically this factorization algebra, call it Λ, contains the determinant bundle as a factorization subbundle. Below we outline the construction and show that this natural factorization algebra is in fact the usual semi-infinite Clifford module vertex algebra.

To define Λ as a factorization algebra on a curve X, we need to assign to each effective divisor D on X a vector space Λ_D such that when D varies, Λ_D becomes a vector bundle (of infinite rank) on the parameter space. Furthermore, we need to exhibit the factorization isomorphisms, i.e., for $D = D_1 + D_2$ with D_1, D_2 having disjoint support, we must naturally identify Λ_D with $\Lambda_{D_1} \otimes \Lambda_{D_2}$.

Fix an effective divisor D, for $n \geq 0$ let

$$W_n = V \otimes \Gamma(X, \mathcal{O}_X(nD)/\mathcal{O}_X(-nD))$$

and

$$W_n^* = V^* \otimes \Gamma(X, \omega_X(nD)/\omega_X(-nD)),$$

where W_n and W_n^* are in fact non-degenerately paired via the residue pairing. Let $V_n = W_n \oplus W_n^*$ with its natural bilinear form $(\cdot\,,\cdot)$. Note that for $m > n$, V_n is naturally a sub-quotient of V_m and denote by $S_{m,n}$ the subspace of V_m

[24] The dependence on multiplicities is eventually eliminated in the limit as is required.
[25] See [10] for instance, where the structure is given by explicit formulas.

that projects onto V_n. Let $K_{m,n}$ be the kernel of this projection and observe that $(K_{m,n}, S_{m,n}) = 0$. Note that

$$A_n = S_{n,0} = V \otimes \Gamma(X, \mathcal{O}_X/\mathcal{O}_X(-nD)) \oplus V^* \otimes \Gamma(X, \omega_X/\omega_X(-nD))$$

is an isotropic subspace of V_n, $A_m \subset S_{m,n}$ projects onto A_n, and $K_{m,n} \subset A_m$. Let

$$\Lambda_n = C(V_n) \otimes_{\wedge A_n} \mathbb{C},$$

where $C(V_n)$ is the Clifford algebra of V_n. Observe that Λ_n is graded by assigning elements of W_n, W_n^* degrees -1 and 1 respectively. Note that by above, for $m > n$, we have $\Lambda_n \hookrightarrow \Lambda_m$ as graded vector spaces. Finally,

$$\Lambda_D := \varinjlim \Lambda_n$$

and one immediately checks that it has all the properties we needed for a factorization structure. Namely, as D varies, V_n, A_n and thus Λ_n form finite dimensional vector bundles on the parameter space. Furthermore, a decomposition of D into disjoint D_1 and D_2 decomposes V_n and A_n into a direct sum, thus Λ_n into a tensor product. Finally, $\Lambda_{sD} = \Lambda_D$ for $s > 0$.

The pullback of Λ to $\mathcal{G}r_{GL(V)}$ naturally contains $\mathcal{L}_{\det}$ as a factorization subbundle. Namely, for a D as above, let $M \in \mathcal{G}r^D_{GL(V)}$, i.e., M is a vector bundle on X equipped with $M|_{X - supp\, D} \cong V \otimes \mathcal{O}_X|_{X-supp\, D}$. Thus for $n \gg 0$,

$$V \otimes \mathcal{O}_X(-nD) \subset M \subset V \otimes \mathcal{O}_X(nD)$$

and denote by L_M the image of $\Gamma(X, M/\mathcal{O}_X(-nD))$ in W_n. Then $L_M \oplus L_M^\perp \subset V_n$ is an isotropic subspace, and let ℓ_M be the line in Λ_n annihilated by L_M. Then the image of ℓ_M in Λ_D is naturally identified with $\mathcal{L}_{\det}|_M$. One immediately sees that the factorization isomorphisms are compatible.

It remains to show that Λ is isomorphic to $\mathcal{C}\ell^\bullet$. First, observe that they are naturally isomorphic as vector spaces by construction. Second, choose a torus $H \subset GL(V)$ and restrict the above factorization compatible map to $\mathcal{G}r_H$, i.e., we have

$$\mathcal{L}_{\det} \to \Lambda \otimes \mathcal{O}_{\mathcal{G}r_H}.$$

Let δ be the D-module of delta functions at every closed point of $\mathcal{G}r_H$. Applying $\Gamma(\mathcal{G}r_H, - \otimes \delta)$ to the above, we obtain a map of factorization algebras on X from a lattice Heisenberg to $\Lambda \otimes \mathcal{O}_{\mathcal{J}\check{H}}$, where $\mathcal{O}_{\mathcal{J}\check{H}}$ is the commutative factorization algebra of functions on the jet scheme of the dual torus. Composing with the restriction to $1 \in \mathcal{J}\check{H}$ we obtain the usual boson-fermion correspondence on the level of vector spaces. Since the map is compatible with factorization structure, we are done. $\square$

Remark. For any κ, there is a natural map of factorization bundles on $\mathcal{G}r_H$:

$$\mathcal{L}_\kappa \to \Gamma(\mathcal{G}r_H, \mathcal{L}_\kappa \otimes \delta) \otimes \mathcal{O}_{\mathcal{G}r_H}$$

obtained by taking the dual of $\mathcal{L}_\kappa^* \leftarrow \Gamma(\mathcal{G}r_H, \mathcal{L}_\kappa^*) \otimes \mathcal{O}_{\mathcal{G}r_H}$. Applying $\Gamma(\mathcal{G}r_H, - \otimes \delta)$ to it, we obtain the co-action map that is the essence of the definition of the lattice Heisenberg according to [6].

Equipped with the above we can proceed.

Proposition 3.4. *As vertex algebras*

$$H^{\infty/2+0}(\mathfrak{n}(\mathcal{K}), A_\kappa(\mathfrak{g})) \cong A_{\kappa-\kappa_c}(\mathfrak{h}).$$

Proof. Consider the diagrams below. On the left i is the inclusion, p the usual projection and a the adjoint action map, on the right are the induced maps on the corresponding Grassmannians:

$$\begin{tikzcd} G & B \arrow[l, hook', "i"'] \arrow[r, "a"] \arrow[d, "p"] & GL(\mathfrak{n}) \\ & H & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G}r_G & \mathcal{G}r_B \arrow[l, hook', "i"'] \arrow[r, "a"] \arrow[d, "p"] & \mathcal{G}r_{GL(\mathfrak{n})} \\ & \mathcal{G}r_H & \end{tikzcd}$$

and everything is compatible with the factorization structure. Call $\mathcal{A}_G$ the D-module on $\mathcal{G}r_G$ corresponding to A under the Satake transform, denote by $\mathcal{A}_H$ the one on $\mathcal{G}r_H$. We have the level bundle $\mathcal{L}_\kappa$ on $\mathcal{G}r_G$, and $\mathcal{L}_{\det}$ the canonical determinant line bundle on $\mathcal{G}r_{GL(\mathfrak{n})}$. Then by the Mirković-Vilonen theorem $p_* i^! \mathcal{A}_G \cong \mathcal{A}_H$. (There are cohomology shifts appearing in the Mirković-Vilonen theorem, but they simply compensate for the modified commutativity constraint. The statement should be interpreted to mean an isomorphism of factorization sheaves.) Thus $\Gamma(\mathcal{G}r_H, p_* i^!(\mathcal{A}_G \otimes \mathcal{L}_\kappa)) \cong A_\kappa(\mathfrak{h})$ as vertex algebras ($\mathcal{L}_\kappa$ is trivialized along the fibers of p).

Note that $a^*\mathcal{L}_{\det}$ is simply the line bundle on $\mathcal{G}r_B$ of relative determinants of the stabilizers in $\mathfrak{n}(\mathcal{K})$ of points in $\mathcal{G}r_B$. Denote also by $\mathcal{C}\ell^\bullet$ the constant bundle on $\mathcal{G}r_B$ with fiber $\bigwedge_{\mathfrak{n}}^\bullet$. As is mentioned above $\mathcal{C}\ell^\bullet$ has factorization structure, furthermore $a^*\mathcal{L}_{\det}$ sits inside as a factorization sub-bundle by Lemma 3.3. We have the following geometric version of the map of Proposition 2.8

$$i.(DR_p^\bullet i^! \mathcal{A}_G \otimes i^* \mathcal{L}_\kappa \otimes a^* \mathcal{L}_{\det}) \longrightarrow i_* i^! \mathcal{A}_G \otimes \mathcal{L}_\kappa \otimes \mathcal{C}\ell^\bullet.$$

It is also compatible with the factorization structure. Applying $\Gamma(\mathcal{G}r_G, -)$ and taking the cohomology gives (by Proposition 3.1) the desired isomorphism

$$A_{\kappa-\kappa_c}(\mathfrak{h}) \xrightarrow{\sim} H^{\infty/2+0}(\mathfrak{n}(\mathcal{K}), \Gamma(\mathcal{G}r_G, i_* i^! \mathcal{A}_G \otimes \mathcal{L}_\kappa)).$$

We observe that the factorization algebra $\mathcal{A}_G \otimes \mathcal{L}_\kappa$ is filtered with the associated graded algebra $i_* i^! \mathcal{A}_G \otimes \mathcal{L}_\kappa$. As before, the $\hat{\mathfrak{h}}_{\kappa-\kappa_c}$ action on the reduction ensures that they have the same vertex algebra structure on their respective cohomologies thus completing the proof. ∎

Denote by $V_{\Gamma,\kappa-\kappa_c}$ the unique up to isomorphism lattice Heisenberg vertex algebra associated to the lattice Γ and the bilinear pairing $(\cdot\,,\cdot)_{\kappa-\kappa_c}$, then we have the following description of the 0th part of the BRST reduction. See Sec. 4.1 for the discussion of the chiral Hecke algebra $S_\kappa(G)$, in particular its description as an explicit vector space.

Corollary 3.5. *As vertex algebras*

$$H^{\infty/2+0}(\mathfrak{n}(\mathcal{K}), S_\kappa(G)) \cong \Gamma(\check{G}/\check{H}, \check{G} \times_{\check{H}} V_{\Gamma,\kappa-\kappa_c}).$$

Proof. By the preceding Theorem we need to describe the vertex algebra $A_{\kappa-\kappa_c}(\mathfrak{h})$ for $A = \mathcal{O}_{\check{G}}$. However as a $\check{H}$-equivariant algebra $\mathcal{O}_{\check{G}} = \Gamma(\check{G}/\check{H}, \check{G} \times_{\check{H}} \mathcal{O}_{\check{H}})$, i.e., it fibers over $\check{G}/\check{H}$ and we note that $A_{\kappa-\kappa_c}(\mathfrak{h})$ for $A = \mathcal{O}_{\check{H}}$ is the lattice Heisenberg vertex algebra $V_{\Gamma,\kappa-\kappa_c}$. Thus for $A = \mathcal{O}_{\check{G}}$, we have that $A_{\kappa-\kappa_c}(\mathfrak{h})$ also fibers over $\check{G}/\check{H}$ and $A_{\kappa-\kappa_c}(\mathfrak{h}) = \Gamma(\check{G}/\check{H}, \check{G} \times_{\check{H}} V_{\Gamma,\kappa-\kappa_c})$. $\square$

Recall that if A has a unit then $V_\kappa(\mathfrak{g}) \subset A_\kappa(\mathfrak{g})$, and so to describe the vertex algebra structure on the reduction of $A_\kappa(\mathfrak{g})$ one must at least understand

$$\Pi := H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), V_\kappa(\mathfrak{g}))$$

as a vertex algebra. It follows from Proposition 3.1 that

$$\Pi \cong \bigoplus_{w \in W} \pi_{w\cdot 0}[-\ell(w)]$$

as a $\hat{\mathfrak{h}}_{\kappa-\kappa_c}$-module. This determines the vertex algebra structure modulo the understanding of multiplication on the highest weights. (At this point a detour through 4.2 is recommended.) These are represented in the cohomology by the cocycles $|w\cdot 0\rangle = v_k \otimes (\omega_w)_0\,|0\rangle$, where v_k and $|0\rangle$ are the generators of $V_\kappa(\mathfrak{g})$ and $\bigwedge_{\mathfrak{n}}^\bullet$ respectively, and ω_w is the cocycle in $\bigwedge^\bullet \mathfrak{n}^*$ that spans $H^\bullet(\mathfrak{n}, \mathbb{C})^{w\cdot 0}$, i.e., $\omega_w = \det(\mathfrak{n}/(\mathfrak{n} \cap \mathfrak{n}_w))^*$. With this in hand one easily computes the leading coefficient (it will occur in degree 0) of the OPE between two highest weight vectors, and obtains the following.

Lemma 3.6. *The highest weight algebra of* Π *is* $H^\bullet(\mathfrak{n}, \mathbb{C})$.

We are now able to completely describe the vertex algebra structure on $H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), S_\kappa(G))$. Recall that

$$\mathcal{V}^\bullet = \bigoplus_{\substack{\lambda \in \Gamma \\ w \in W}} \mathbb{C}_{-\lambda} \otimes \pi_{w\cdot 0-(\kappa-\kappa_c)\lambda}[-\ell(w)]$$

can be given the structure of a vertex algebra as described in Sec. 4.2.

Theorem 3.7. *As a vertex algebra*

$$H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), S_\kappa(G)) \cong \Gamma(\check{G}/\check{H}, \check{G} \times_{\check{H}} \mathcal{V}^\bullet).$$

Remark. We note that while the above Theorem addresses the BRST reduction of the untwisted chiral Hecke algebra $S_\kappa(G)$, it is readily applied to the twisted case. More precisely, recall that $S_\kappa(G)_\phi$ denotes the twist of $S_\kappa(G)$ by a $\check{G}$-local system ϕ on $\mathrm{Spec}(\mathcal{K})$. Then

$$H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), S_\kappa(G)_\phi) \cong \Gamma(\check{G}/\check{H}, \check{G} \times_{\check{H}} \mathcal{V}^\bullet)_\phi$$

where the right-hand side denotes the twist of $\Gamma(\check{G}/\check{H}, \check{G} \times_{\check{H}} \mathcal{V}^\bullet)$ by a $\check{G}$-local system ϕ.

Proof. By Corollary 3.5 and the remark in 4.2, we see that $\Gamma(\check{G}/\check{H}, \mathcal{O}_{\check{G}/\check{H}})$ is central in $\mathcal{H} := H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), S_\kappa(G))$. Thus we can realize this vertex algebra as global sections of a sheaf of vertex algebras over $\check{G}/\check{H}$. Recall that Corollary 3.5 identifies $\mathcal{H}_0 := H^{\infty/2+0}(\mathfrak{n}(\mathcal{K}), S_\kappa(G))$ as a vertex algebra with $\Gamma(\check{G}/\check{H}, \check{G} \times_{\check{H}} V_{\Gamma,\kappa-\kappa_c})$, and consider for every $w \in W$ the $\mathcal{H}_0$-submodule of $\mathcal{H}$ generated by $|w \cdot 0\rangle$; denote it by $\mathcal{H}_w$.

Irreducible representations of $V_{\Gamma,\kappa-\kappa_c}$ are parameterized by $\check{\Gamma}/(\kappa-\kappa_c)(\Gamma)$ (see for example [10]). More precisely, if $\alpha \in \check{\Gamma}$, then the irreducible representation indexed by $\overline{\alpha} \in \check{\Gamma}/(\kappa-\kappa_c)(\Gamma)$, let us call it the highest weight representation of highest weight α and denote it by U_α, is given as a $\hat{\mathfrak{h}}_{\kappa-\kappa_c}$-module by $\bigoplus_{\lambda\in\Gamma} \pi_{\alpha-(\kappa-\kappa_c)\lambda}$; it is generated as a $V_{\Gamma,\kappa-\kappa_c}$-module by $|\alpha\rangle$. Observe that by considering α instead of $\overline{\alpha}$, U_α is naturally a $\check{H}$-module, i.e., the $\check{H}$-equivariant irreducible representations of $V_{\Gamma,\kappa-\kappa_c}$ are indexed by $\check{\Gamma}$ itself (more precisely, $|\alpha\rangle$ is the highest weight of the π_0-module $U_\alpha^{\check{H}}$). We note that for κ sufficiently negative in our sense, all the $\overline{w \cdot 0} \in \check{\Gamma}/(\kappa-\kappa_c)(\Gamma)$ are distinct for different $w \in W$, thus indexing non-isomorphic representations of $V_{\Gamma,\kappa-\kappa_c}$.

Thus, using Corollary 3.2, $\mathcal{H}_w$ can be identified with $\Gamma(\check{G}/\check{H}, \check{G} \times_{\check{H}} U_{w\cdot 0})$, and $\mathcal{H} = \bigoplus_{w\in W} \mathcal{H}_w$ as an $\mathcal{H}_0$-module. Let $A^\lambda = \bigoplus_{\chi\in\Gamma^+} V_\chi^* \otimes V_\chi(\lambda)$ so that $\mathcal{H} = \bigoplus_{\lambda,w} A^\lambda \otimes \pi_{w\cdot 0-(\kappa-\kappa_c)\lambda}$. Then

$$hwa(\mathcal{H}) = \bigoplus_{\lambda,w} A^\lambda \otimes |\lambda + w \cdot 0\rangle\,,$$

where we retain the $|\lambda + w\cdot 0\rangle$ to keep track of the a priori different A^λ. Knowledge of $\mathcal{H}$ as an $\mathcal{H}_0$-module allows us to compute (for $a_\lambda \in A^\lambda$, $a_\chi \in A^\chi$ and $w, w' \in W$ such that $\omega_w \cdot \omega_{w'} = \pm\omega_{w''}$):

$$\begin{aligned}
a_\lambda \otimes & |\lambda + w\cdot 0\rangle \cdot a_\chi \otimes |\chi + w'\cdot 0\rangle \\
&= a_\lambda \otimes |\lambda\rangle \cdot 1 \otimes |w\cdot 0\rangle \cdot a_\chi \otimes |\chi\rangle \cdot 1 \otimes |w'\cdot 0\rangle \\
&= (-1)^{\ell(w)(\chi,\chi)+w\cdot 0(\chi)} a_\lambda \otimes |\lambda\rangle \cdot a_\chi \otimes |\chi\rangle \cdot 1 \otimes |w\cdot 0\rangle \cdot 1 \otimes |w'\cdot 0\rangle \\
&= \pm(-1)^{\ell(w)(\chi,\chi)+w\cdot 0(\chi)} a_\lambda a_\chi \otimes |\lambda+\chi\rangle \cdot 1 \otimes |w''\cdot 0\rangle \\
&= \pm(-1)^{\ell(w)(\chi,\chi)+w\cdot 0(\chi)} a_\lambda a_\chi \otimes |\lambda+\chi+w''\cdot 0\rangle\,.
\end{aligned}$$

We conclude that $hwa(\mathcal{H}) \cong hwa(\mathcal{H}_0) \widetilde{\otimes} H^\bullet(\mathfrak{n}, \mathbb{C})$ and the claim follows.

$\square$

As was mentioned in the introduction, the unramified case of the geometric local Langlands correspondence manifests itself in our situation in the form of the D-modules on the affine flags that we called monodromy annihilators. Recall that such a D-module M has the property that the monodromy action on $\mathcal{Z}(V)$, with V any representation of $\check{G}$, becomes trivial on $\mathcal{Z}(V) \star M$. The importance of this notion for us is that these M provide $\check{G}$-equivariant representations of the untwisted chiral Hecke algebra $S_\kappa(G)$ via $\Gamma(\mathcal{F}\ell, (\mathcal{Z}(\mathcal{O}_{\check{G}}) \star -) \otimes \mathcal{L}_{\kappa+\chi})$. In fact, conjecturally, these are all of them.

In particular, D-modules pulled back to the affine flags from the affine Grassmannian are in some sense the most important examples of the monodromy annihilators.[26] To obtain a series of $(S_\kappa(G), \check{G})$-modules from them one need not even leave the affine Grassmannian. Recall that for a D-module M on $\mathcal{G}r$, we have that $\Gamma(\mathcal{G}r, (\widetilde{\mathcal{O}}_{\check{G}} \star M) \otimes \mathcal{L}_\kappa)$ is an $(S_\kappa(G), \check{G})$-module.[27] We would like to consider its BRST reduction and describe it as a module over the BRST reduction of $S_\kappa(G)$ itself.

It follows from Theorem 3.7 that the BRST reduction of a $\check{G}$-equivariant $S_\kappa(G)$-module V fibers equivariantly over $\check{G}/\check{H}$ so that it is completely determined by the structure of the fiber over $1 \in \check{G}/\check{H}$, let us denote it by $\mathcal{B}(V)$, as a $\check{H} \times \mathbb{G}_m$-equivariant $\mathcal{V}^\bullet$-module. This itself is determined by the structure of $\mathcal{B}(V)^{\check{H}}$ as a $\mathbb{G}_m$-equivariant, i.e. graded, Π-module. In the case under consideration

$$V = \Gamma(\mathcal{G}r, (\widetilde{\mathcal{O}}_{\check{G}} \star M) \otimes \mathcal{L}_\kappa)$$

and

$$\mathcal{B}(V)^{\check{H}} = H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), \Gamma(\mathcal{G}r, M \otimes \mathcal{L}_\kappa)).$$

The latter can be computed as a π_0-module using the Remark that follows Proposition 3.1, and as in the proof of Lemma 3.6 we see that the action of the whole Π is "free". Consequently we obtain the following Corollary.

Corollary 3.8. *Let M be a D-module on $\mathcal{G}r$ and*

$$\mathcal{H}(M) := \bigoplus_{\chi \in \Gamma} H^\bullet_{DR}(S_\chi, i^!_\chi M)(\chi)[-2ht\chi]$$

the associated $\check{H} \times \mathbb{G}_m$-module, then as $(\mathcal{V}^\bullet, \check{H} \times \mathbb{G}_m)$-modules

$$\mathcal{B}(\Gamma(\mathcal{G}r, (\widetilde{\mathcal{O}}_{\check{G}} \star M) \otimes \mathcal{L}_\kappa)) \cong \mathcal{V}^\bullet \otimes \mathcal{H}(M)$$

[26] Their importance, conjectural and otherwise, is discussed in the introduction.

[27] This is the same $(S_\kappa(G), \check{G})$-module as the one obtained via pullback to $\mathcal{F}\ell$, i.e. it is isomorphic to $\Gamma(\mathcal{F}\ell, (\mathcal{Z}(\mathcal{O}_{\check{G}}) \star \pi^* M) \otimes \mathcal{L}_{\kappa+2\rho})$. Note that in order to obtain all of the $(S_\kappa(G), \check{G})$-modules that come from $\mathcal{G}r$ one does need to pull back to $\mathcal{F}\ell$ first as otherwise any twist other than by 2ρ is unavailable.

where $\mathcal{V}^\bullet$ is viewed as a $\check{H} \times \mathbb{G}_m$-equivariant module over itself and so can be twisted by the $\check{H} \times \mathbb{G}_m$-module $\mathcal{H}(M)$.

One may thus conjecture that the $\check{G}$-equivariant $S_\kappa(G)$-modules that arise as $V = \Gamma(\mathcal{G}r, (\widetilde{\mathcal{O}}_{\check{G}} \star M) \otimes \mathcal{L}_\kappa)$ are characterized by the property that $\mathcal{B}(V^g)$ is of the form $\mathcal{V}^\bullet \otimes \mathcal{M}_g$ for every $g \in G(\mathcal{K})$, where $\mathcal{M}_g$ is some $\check{H} \times \mathbb{G}_m$-module. By interpreting $\mathcal{M}_g$ as $\mathcal{H}(g^* M)$ for some D-module M on $\mathcal{G}r$ one should be able to recover M itself.

The BRST reduction of other series of $(S_\kappa(G), \check{G})$-modules that come from $\mathcal{G}r$, i.e. those arising from twisting by a character other than 2ρ, can be similarly described through the structure of their fibres over $1 \in \check{G}/\check{H}$ as $\check{H} \times \mathbb{G}_m$-equivariant $\mathcal{V}^\bullet$-modules. They are still "free", though now modeled not on $\mathcal{V}^\bullet$ itself, but rather on a shift of it. This is similar and in fact caused by, a similar phenomenon that occurs for lattice Heisenberg modules; they include the lattice Heisenberg itself and a finite number of its shifts.

The situation for other monodromy annihilators on $\mathcal{F}\ell$, i.e those that do not arise as pullbacks from $\mathcal{G}r$, is more complicated. After applying the BRST reduction functor one can still restrict to the fiber over $1 \in \check{G}/\check{H}$, however the resulting module over $\mathcal{V}^\bullet$ is no longer "free" and is in general rather arbitrary. However, if we restrict our attention only to the $V_{\Gamma, \kappa - \kappa_c}$-module[28] structure, then the situation is again very manageable. Namely, recall that the irreducible $(V_{\Gamma, \kappa-\kappa_c}, \check{H})$-module U_α is characterized by the property that $U_\alpha^{\check{H}} \cong \pi_\alpha$ as a π_0-module. As before we have that

$$\mathcal{B}(\Gamma(\mathcal{F}\ell, (\mathcal{Z}(\mathcal{O}_{\check{G}}) \star M) \otimes \mathcal{L}_{\kappa+\chi}))^{\check{H}} = H^{\infty/2+\bullet}(\mathfrak{n}(\mathcal{K}), \Gamma(\mathcal{F}\ell, M \otimes \mathcal{L}_{\kappa+\chi}))$$

and the latter can be computed as a π_0-module using Theorem 2.9. Let us summarize as follows.

Corollary 3.9. *Let M be a monodromy annihilator D-module on $\mathcal{F}\ell$, and set $V = \Gamma(\mathcal{F}\ell, (\mathcal{Z}(\mathcal{O}_{\check{G}}) \star M) \otimes \mathcal{L}_{\kappa+\chi})$, then as $(V_{\Gamma,\kappa-\kappa_c}, \check{H} \times \mathbb{G}_m)$-modules*

$$\mathcal{B}(V) \cong \bigoplus_{w \in W_{\mathit{aff}}} U_{w\cdot(-\chi)} \otimes H^\bullet_{DR}(S_w, i^!_w M)[-2ht(\lambda_w) - \ell(\bar{w})]$$

where $w = \lambda_w \bar{w}$.

4 Appendix.

Here we collect some auxiliary information that we hope will make the paper more accessible to the reader.

[28] Recall that $V_{\Gamma,\kappa-\kappa_c} = \mathcal{V}^0 \subset \mathcal{V}^\bullet$.

4.1 The chiral Hecke algebra.

The chiral Hecke algebra, introduced by Beilinson-Drinfeld, is defined using the geometric version of the Satake isomorphism [19, 21], which is an equivalence (of tensor categories) between the category of representations of the Langlands dual group $\check{G}$ and the graded (by dimension of support) category of $G(\mathcal{O})$-equivariant D-modules on the affine Grassmannian (here the tensor structure is given by convolution). The functor from D-modules to $\check{G}$ representations is just $H^{\bullet}_{DR}(\mathcal{G}r, -)$. Under this equivalence a commutative algebra structure on any $\check{G}$-module produces a chiral algebra structure on the $\mathcal{L}_\kappa$-twisted global sections (κ chosen negative integral) of the corresponding D-module as follows.

Let A be a commutative algebra and a $\check{G}$-module such that the multiplication $m : A \otimes A \to A$ is a map of $\check{G}$-modules. Let $\mathcal{A}$ be the corresponding (under the Satake isomorphism) $G(\mathcal{O})$-equivariant D-module on $\mathcal{G}r$, and $\tilde{m} : \mathcal{A} * \mathcal{A} \to \mathcal{A}$ the corresponding map of D-modules. Let X be a curve and consider the diagram (Δ is the embedding of the diagonal, j of the complement):

$$\begin{tikzcd}
\mathcal{G}r_X \arrow[r, hook, "\Delta"] \arrow[d, "p"] & \mathcal{G}r_X^{(2)} \arrow[d, "p"] & \mathcal{G}r_X \times \mathcal{G}r_X|_U \arrow[l, hook', "j"'] \arrow[d, "p"] \\
X \arrow[r, hook, "\Delta"] & X \times X & U \arrow[l, hook', "j"']
\end{tikzcd}$$

One of the definitions of $\mathcal{A} * \mathcal{A}$ is as $\Delta^! j_{!*}(\mathcal{A} \boxtimes \mathcal{A})|_U[1]$, and so we get the diagram below:

$$\begin{tikzcd}
0 \arrow[r] & j_{!*}(\mathcal{A} \boxtimes \mathcal{A})|_U \arrow[r] & j_*(\mathcal{A} \boxtimes \mathcal{A})|_U \arrow[r] \arrow[dr, "\alpha"] & \Delta_*(\mathcal{A} * \mathcal{A}) \arrow[r] \arrow[d, "\Delta_* \tilde{m}"] & 0 \\
& & & \Delta_* \mathcal{A} &
\end{tikzcd}$$

On $\mathcal{G}r_X^{(2)}$, we have $\mathcal{L}_\kappa^{(2)}$ providing the factorization structure on the level bundle $\mathcal{L}_\kappa$. When we twist the morphism α in the diagram above by $\mathcal{L}_\kappa^{(2)}$, we obtain

$$j_*(\mathcal{A} \otimes \mathcal{L}_\kappa \boxtimes \mathcal{A} \otimes \mathcal{L}_\kappa)|_U \longrightarrow \Delta_*(\mathcal{A} \otimes \mathcal{L}_\kappa).$$

By applying $p.$ to the above, which is exact, we get a chiral bracket on $\Gamma(\mathcal{G}r, \mathcal{A} \otimes \mathcal{L}_\kappa)$. Note the use of $\mathcal{A}$ for both the D-module on $\mathcal{G}r$ and also on $\mathcal{G}r_X$. We denote by $p.$ the direct image functor on the category of $\mathcal{O}$-modules, to be contrasted with p_* playing the same role for the category of D-modules.

Remark. It is worthwhile to note that if instead of $p.$ above, we apply p_*, necessarily to the untwisted version of the diagram, then we again obtain a chiral bracket, on A this time, which can be constructed in a standard way from the commutative associative product on A. Namely, in the vertex algebra language $Y(a, z) = L_a$,

for $a \in A$ and L_a denoting the left multiplication operator. Alternatively, in the chiral algebra language, the multiplication on A, gives a morphism of D-modules on X

$$A^r \otimes^! A^r \to A^r$$

where $A^r = A \otimes_{\mathbb{C}} \Omega_X$ and $\otimes^!$ denotes the tensor product of right D-modules obtained from the standard $\otimes$ on left D-modules via the usual right-left identification. The chiral bracket is then constructed in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & A^r \boxtimes A^r \arrow[r] & j_*j^*A^r \boxtimes A^r \arrow[r] \arrow[dr] & \Delta_*(A^r \otimes^! A^r) \arrow[r] \arrow[d] & 0 \\
& & & \Delta_* A^r &
\end{tikzcd}$$

Due to the nature of the commutativity constraint, the chiral algebra we obtain is graded. One way to describe the grading is to say that the component arising via the Satake isomorphism from the $\check{G}$-module V_χ has parity $(\chi, \chi)_{\kappa-\kappa_c}$ mod 2.

This procedure applied to the trivial representation yields the Kac-Moody vertex algebra $V_\kappa(\mathfrak{g})$, while the regular representation produces the chiral Hecke algebra $S_\kappa(G)$:

$$S_\kappa(G) = \bigoplus_{\chi \in \Gamma^+} V_\chi^* \otimes \Gamma(\mathcal{G}r, I_\chi \otimes \mathcal{L}_\kappa)$$

where $I_\chi = i_{!*}\Omega_\chi$ the standard $G(\mathcal{O})$-equivariant D-module supported on $\mathcal{G}r^\chi$. As is mentioned above, the parity of $V_\chi^* \otimes \Gamma(\mathcal{G}r, I_\chi \otimes \mathcal{L}_\kappa)$ is $(\chi, \chi)_{\kappa-\kappa_c}$ mod 2.

Remark. If $G = H$, i.e., G is a torus, then $S_\kappa(G) = V_{\Gamma,\kappa}$, the lattice Heisenberg vertex algebra. Its representation theory, in this case, is well understood, and so the conjecture described in the introduction is quite obviously true.

4.2 Highest weight algebras.

The purpose of this section is to explain precisely how a $\hat{\mathfrak{h}}_\kappa$-module structure on a vertex algebra essentially determines it, the remaining information is encoded in what we call the highest weight algebra (which is an example of a twisted commutative algebra). Related notions, necessary for our purposes are discussed. The notation is borrowed from [10].

Definition 4.1. A twisted commutative algebra A is first of all a Γ_A-graded unital associative super-algebra, where Γ_A is a lattice and the parity is given by a $p : \Gamma_A \to \mathbb{Z}_2$. We also require the additional structure of a symmetric bilinear pairing $(\cdot\,,\cdot) : \Gamma_A \otimes \Gamma_A \to \mathbb{Q}$, with $(\lambda, \chi) \in \mathbb{Z}$ if $S_{\lambda,\chi} \neq 0$, where $S_{\lambda,\chi} : A^\lambda \otimes A^\chi \to A^{\lambda+\chi}$ denotes the restriction of the multiplication in A (so that $(\cdot\,,\cdot)$ is essentially integral). Finally A must satisfy a $(\cdot\,,\cdot)$-twisted commutativity constraint, i.e the following diagram must commute (if $S_{\lambda,\chi} \neq 0$):

$$
\begin{tikzcd}
a \otimes b \arrow[d, mapsto, "\sigma"'] & & A^\lambda \otimes A^\chi \arrow[r, "S_{\lambda,\chi}"] \arrow[d, "\sigma"'] & A^{\lambda+\chi} \arrow[d, equal] \\
(-1)^{p(\lambda)p(\chi)+(\lambda,\chi)} b \otimes a & & A^\chi \otimes A^\lambda \arrow[r, "S_{\chi,\lambda}"] & A^{\lambda+\chi}
\end{tikzcd}
$$

and we note that the commutativity constraint forces a certain compatibility between $(\cdot\,,\cdot)$ and p, namely if $S_{\lambda,\lambda} \neq 0$ then $p(\lambda) = (\lambda, \lambda) \bmod 2$, else it is extra data.

Consider a $\hat{\mathfrak{h}}_\kappa$-module and conformal (super) vertex algebra

$$V = \bigoplus_{\lambda \in \Gamma_V} A^\lambda \otimes \pi_\lambda$$

where the lattice Γ_V comes with a map of abelian groups to $\mathfrak{h}^*$ (so that we may treat the lattice points as elements of $\mathfrak{h}^*$). We assume that A^λ are finite dimensional vector spaces and the action of $\hat{\mathfrak{h}}_\kappa$ is trivially extended to $A^\lambda \otimes \pi_\lambda$ from π_λ. Recall that π_λ is the Fock representation of $\hat{\mathfrak{h}}_\kappa$, i.e., it is the module generated by the highest weight vector $|\lambda\rangle$ subject to $h_n |\lambda\rangle = 0$ if $n > 0$ and $h_0 |\lambda\rangle = \lambda(h_0) |\lambda\rangle$, where $h \in \mathfrak{h}$ and $h_n = h \otimes t^n$.

Suppose that $\pi_0 \subset V$ (which is itself a Heisenberg vertex algebra associated to the Heisenberg Lie algebra $\hat{\mathfrak{h}}_\kappa$) is a vertex subalgebra of V (we identify π_0 with $\pi_0 \cdot 1_V \subset V$), whose action on V is compatible with that of $\hat{\mathfrak{h}}_\kappa$. Let $a_\lambda \in A^\lambda$ and denote by $V_{a_\lambda}(w)$ the field $Y(a_\lambda \otimes |\lambda\rangle, w)$ associated to $a_\lambda \otimes |\lambda\rangle \in A^\lambda \otimes \pi_\lambda$. Then these fields completely determine the vertex algebra structure of V. But an explicit computation (essentially present in [10], explicitly in [23]) shows that the fields themselves are determined up to the operations

$$S_{\lambda,\chi} : A^\lambda \otimes A^\chi \to A^{\lambda+\chi}$$

on $A = \bigoplus_{\Gamma_V} A^\lambda$ obtained as follows: $S_{\lambda,\chi}(a_\lambda, a_\chi) \in A^{\lambda+\chi}$ is the leading coefficient of the series $Y(a_\lambda \otimes |\lambda\rangle, w)(a_\chi \otimes |\chi\rangle)$ in $A^{\lambda+\chi}((w))$. Note that there is a distinguished element $1_A \in A^0$ obtained via $1_A \otimes |0\rangle = 1_V$.

Definition 4.2. We call $A = \bigoplus_{\Gamma_V} A^\lambda$ with the operations $S_{\lambda,\chi}$ the highest weight algebra of V and denote it $hwa(V)$.

Remark. Note that the commutative algebra $A^0 \otimes |0\rangle \subset V$ is in the center of V.

More precisely, let $\bar{\lambda}$ denote the image in $\mathfrak{h}$ of λ under κ (we use κ to denote the isomorphism induced by $(\cdot\,,\cdot)_\kappa$). For $h \in \mathfrak{h}$ let $b^h(w)_- = \sum_{n<0} h_n w^{-n-1}$ and $b^h(w)_+ = \sum_{n>0} h_n w^{-n-1}$ then we have the following Lemma.

Lemma 4.3. *With V as above*

$$V_{a_\lambda}(w) = S_{\lambda,\bullet}(a_\lambda, -) \otimes w^{\bullet(\bar{\lambda})} e^{\int b^{\bar{\lambda}}(w)_-} e^{\int b^{\bar{\lambda}}(w)_+}$$

and $hwa(V)$ is a twisted commutative algebra. The lattice is Γ_V, the parity is inherited from V, and the pairing is given by $(\lambda, \chi) = \chi(\bar{\lambda})$.

Remark. By starting with a twisted commutative algebra $A = \bigoplus_{\lambda \in \Gamma_A} A^\lambda$ and equipped with a homomorphism $\psi : \Gamma_A \to \mathfrak{h}^*$, subject to the compatibility condition $(\lambda, \chi) = \psi(\chi)(\overline{\psi(\lambda)})$, we can define a vertex algebra structure on the $\hat{\mathfrak{h}}_\kappa$-module $\bigoplus A^\lambda \otimes \pi_\lambda$ via the formula in Lemma 4.3.

One can describe the lattice Heisenberg vertex algebra via this approach, namely its highest weight algebra is constructed as follows. Consider a commutative (forgetting the grading) algebra A together with a Γ_A grading, and p, $(\cdot\,,\cdot)$ as above. Then assuming that $p(\lambda) = (\lambda, \lambda) \bmod 2$, we can modify the multiplication on A to get $\widetilde{A}$, a $(\cdot\,,\cdot)$-twisted commutative algebra. This procedure is very similar to the one described in [6]. Let us begin by choosing an ordered basis $\mathcal{B}$ of Γ_A. For $\lambda, \chi \in \mathcal{B}$, define

$$r(\lambda, \chi) = \begin{cases} p(\lambda)p(\chi) + (\lambda, \chi) & \lambda > \chi \\ 0 & \text{else} \end{cases}$$

and extend to Γ_A by linearity. Then if $S_{\lambda,\chi} : A^\lambda \otimes A^\chi \to A^{\lambda+\chi}$, let

$$\widetilde{S}_{\lambda,\chi} = (-1)^{r(\lambda,\chi)} S_{\lambda,\chi}.$$

This gives $\widetilde{A}$ the required twisted commutative algebra structure. For the lattice Heisenberg vertex algebra we start with the commutative algebra $\mathbb{C}\Gamma$. In our case Γ is the co-weight lattice and the level (in our case $\kappa - \kappa_c$) is the pairing $(\cdot\,,\cdot)$. The resulting twisted commutative algebra $\widetilde{\mathbb{C}\Gamma}$ is the highest weight algebra of $V_{\Gamma,\kappa-\kappa_c}$.

Definition 4.4. Given two twisted commutative algebras A and B, together with a bilinear pairing $(\cdot\,,\cdot) : \Gamma \otimes \Gamma \to \mathbb{Z}$ ($\Gamma = \Gamma_A \oplus \Gamma_B$) extending[29] those on Γ_A and Γ_B, we can form the twisted tensor product $A \widetilde{\otimes} B$, again a twisted commutative algebra, by letting

$$a \otimes b \cdot a' \otimes b' = (-1)^{p(\lambda)p(\chi) + (\lambda,\chi)} a \cdot a' \otimes b \cdot b'$$

for $b \in B^\lambda$ and $a' \in A^\chi$.

The statement of Theorem 3.7 requires three things from this section. First we need the twisted commutative algebra obtained from the lattice Heisenberg vertex algebra, it is described above. This is a non-degenerate example in the sense that all $S_{\lambda,\chi}$ are non-0. In fact this non-degeneracy alone implies that up to isomorphism it is a lattice Heisenberg vertex algebra.

Our second example is $H^\bullet(\mathfrak{n}, \mathbb{C})$, a very degenerate case, namely we take as our lattice the weight lattice (the only non-0 components are the lines at $w \cdot 0$ for $w \in W$). The pairing $(\cdot\,,\cdot)$ is $(\kappa - \kappa_c)^{-1}$. Note that whenever the product of

[29] In the case that is of interest to us, this extension is not the trivial one.

two elements of $H^\bullet(\mathfrak{n}, \mathbb{C})$ is non-0, their weights are orthogonal with respect to $(\kappa - \kappa_c)^{-1}$, so this does not conflict with the essential integrality of $(\cdot\,,\cdot)$. The parity is given by the cohomological degree modulo 2. We note that the triviality of $(\cdot\,,\cdot)$ is necessary because $H^\bullet(\mathfrak{n}, \mathbb{C})$ is super-commutative. This example comes up in Lemma 3.6.

Finally the twisted commutative algebra that we need in Theorem 3.7 is formed by taking the twisted tensor product of the two examples above. The extension of the pairing to the direct sum of the weight and the co-weight lattices is done through their natural pairing. More precisely, $(w \cdot 0, \chi) = w \cdot 0(\chi)$, i.e., it is truly a twisted product. We call the resulting vertex algebra $\mathcal{V}^\bullet$, thus

$$hwa(\mathcal{V}^\bullet) \cong \widetilde{\mathbb{C}\Gamma} \widetilde{\otimes} H^\bullet(\mathfrak{n}, \mathbb{C}).$$

References

[1] S. Arkhipov, R. Bezrukavnikov, *Perverse sheaves on affine flags and Langlands dual group*, Preprint math.RT/0201073.

[2] A. Beilinson, *Langlands parameters for Heisenberg modules*, Preprint math.QA/0204020.

[3] A. Beilinson, J. Bernstein, *Localisation de $\mathfrak{g}$-modules*, C.R. Acad. Sci. Paris, Sér. I Math. **292** (1981), no. 1, 15–18.

[4] A. Beilinson, J. Bernstein, *A generalization of Casselman's submodule theorem*, Representation theory of reductive groups (Park City, Utah, 1982) **40** (1983), no. 1, 35–52.

[5] A. Beilinson, V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, Preprint, available at www.math.uchicago.edu/∼mitya

[6] A. Beilinson, V. Drinfeld, *Chiral Algebras*, American Mathematical Society Colloquium Publications **51**, AMS, 2004

[7] R. Bezrukavnikov, *Perverse sheaves on affine flags and nilpotent cone of the Langlands dual group*, Preprint math.RT/0201256.

[8] J. Brylinski, M. Kashiwara, *Dmonstration de la conjecture de Kazhdan-Lusztig sur les modules de Verma*, (French) C. R. Acad. Sci. Paris Sr. A-B **291** (1980), no. 6, A373–A376.

[9] J. Brylinski, M. Kashiwara, *Kazhdan-Lusztig conjecture and holonomic systems*, Invent. Math. **64** (1981), 387–410.

[10] E. Frenkel, D. Ben-Zvi, *Vertex algebras and Algebraic Curves.* Second edition, Mathematical Surveys and Monographs **88**, AMS, 2004

[11] E. Frenkel, *Langlands correspondence for loop groups.* Cambridge Studies in Advanced Mathematics, **103**. Cambridge University Press, Cambridge, 2007.

[12] E. Frenkel, D. Gaitsgory, *D-modules on the affine Grassmannian and representations of affine Kac-Moody algebras*, Preprint math.AG/0303173.

[13] E. Frenkel, D. Gaitsgory, *Fusion and convolution: applications to affine Kac-Moody algebras at the critical level*, Preprint math.RT/0511284.

[14] E. Frenkel, D. Gaitsgory, K. Vilonen, *Whittaker Patterns in the Geometry of Moduli Spaces of Bundles on Curves*, Annals of Mathematics, **153** (2001), 699–748.

[15] E. Frenkel, D. Gaitsgory, *D-modules on the affine flag variety and representations of affine Kac-Moody algebras*, Preprint math.RT/0712.0788.

[16] D. Gaitsgory, *Notes on 2D conformal field theory and string theory.* Quantum fields and strings: a course for mathematicians, Vol. 2 (Princeton, NJ, 1996/1997), 1017–1089, Amer. Math. Soc., Providence, RI, 1999.

[17] D. Gaitsgory, *The notion of category over an algebraic stack*, Preprint math.AG/0507192.

[18] D. Gaitsgory, *Construction of central elements in the affine Hecke algebra via nearby cycles*, Preprint math.AG/9912074.

[19] V. Ginzburg, *Perverse sheaves on a Loop group and Langlands' duality*, Preprint math.AG/9511007.

[20] V. Kac, D. Kazhdan, *Structure of representations with highest weight of infinite-dimensional Lie algebras*, Adv. in Math. **34** (1979), no. 1, 97–108.

[21] I. Mirković, K. Vilonen, *Perverse Sheaves on affine Grassmannians and Langlands Duality*, Preprint math.AG/9911050.

[22] I. Mirković, K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Preprint math.RT/0401222.

[23] I. Shapiro, *BRST reduction of the chiral Hecke algebra*, Ph. D. Thesis, University of Chicago (2004)

[24] A. Voronov, *Semi-Infinite Induction and Wakimoto Modules*, Preprint q-alg/9704020.

[25] A. Voronov, *Semi-infinite homological algebra*, Invent. Math. **113** (1993), no. 1, 103–146.

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